SIX LECTURES
ON
THE MEAN-VALUE THEOREM
OF
THE DIFFERENTIAL CALCULUS

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OF
THE DIFFERENTIAL CALCULUS

delivered at the Calcutta University

BY

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PREFACE

In this book are contained without any material alteration the six public lectures, which were delivered by me during the first four months of 1930 with the two-fold object of (1) giving an up-to-date account of our knowledge of the mean-value theorem and the function θ to which I have ventured to give the name of Rolle's function, and (2) stimulating research relating to Rolle's function by suggesting problems which were at the time engaging my attention and had not been completely solved.

Apart from the first chapter, which is historical and introductory, the book may be roughly divided into two parts, *viz.*, the second and third chapters which deal with the mean-value theorem and its generalizations, and the fourth and fifth chapters and the major part of the sixth chapter in which Rolle's function has been studied. The first Appendix is of interest because of the correspondence between Professor Pompeiu and myself about his remarkable proof of the mean-value theorem. The second Appendix deals with the history of the various forms of the remainder in Taylor's series. The third Appendix contains corrections and additions.

I venture to say that the chief interest of the book is the prominent place given in it to Rolle's function. It is true that the function was considered by Cauchy about 100 years ago and was later studied by American and Cambridge mathematicians without more being discovered about its functional property than that it may be multiple-valued. It is only very recently that well-known mathematicians, for example, Professor Rudolf Rothe and Professor T. Hayashi, took up the study of the function but in ignorance of much of the earlier work. A few of the landmarks in this field of research may be enumerated here. (1) Professor Rothe had never contemplated the possibility of $\theta(h)$ being non-differentiable; I have given functions $\theta(h)$ which are single-valued, finite and continuous and at the same time without differential co-efficients at the points of an everywhere dense set. (2) Prof. Hedrick had attempted the study of $\theta(h)$ as a multiple-valued function, but, for want of the separate treatment of the different values corresponding to a given h, his treatment is confused and infructuous. I have introduced the notion of *principal value* of $\theta(h)$ as the greatest of the different values and shown

that, for $f'(h)$ as nowhere differentiable, $\theta(h)$ may be also nowhere differentiable, with the possible exception of the points of the first category where $f'(h)$ has "cusps."

It is a great pleasure to me to record my obligations to a number of my friends and pupils who have helped me in various ways, during the time the book was in the Press. To Dr. Bibhutibhushan Datta, D.Sc., I am indebted for his advice relating to the parts which are of historical character. Dr. A. N. Singh, D.Sc., Lecturer in the Lucknow University, has gone through nearly the whole of the book in proof and has given me valuable advice relating to certain parts of it. Mr. Hariprasanna Banerjee, M.Sc., Lecturer in Pure Mathematics to the Post-graduate students in the Calcutta University, has also helped me like Dr. Singh with valuable advice. Some of my present researchers, specially Mr. Bholanath Mookerjee, M.A., P.R.S., of Scottish Church College, Mr. Santoshkumar Bhar, M.Sc., and Mr. Rama Dhar Misra, M.A., have also given me help in going through the proof-sheets.

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GANESH PRASAD

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FIRST LECTURE

Historical and Introductory.

Colleagues, students and other gentlemen!

The six lectures which I propose to deliver on the mean-value theorem, rightly called the fundamental theorem of the Differential Calculus, are intended to make known to you a field of research, which has only in comparatively recent times attracted anything like considerable attention from prominent mathematicians and which is far from being exhausted. I shall feel amply rewarded if I succeed in inspiring some of you with enthusiasm to take up for research even a few of the numerous problems that may be suggested by my lectures.

§ 1.

1. To-day's lecture will be of a historical and introductory character, and I shall begin by giving you an account of the vicissitudes through which Rolle's theorem, from which the mean-value theorem originated, has gone. Michel Rolle (1652-1719), who was from 1685 onwards a paid member of the Academy of Sciences of Paris, was one of the small band of the "old mathematicians who did not content themselves with the intuitional proofs of theorems but demanded rigorous logical definitions and proofs."[1] In a small duodecimo book,[2] published in 1691, he gave what goes now by the name of Rolle's theorem,[3] viz., $f'(x)=0$ has at least one

[1] See Felix Klein's *Anwendung der Differential und Integral-rechnung auf Geometrie*, 1st edition, p. 114.

[2] *Démonstration d'une Méthode pour résoudre les Egalitez de tous les degrez ; suivie de deux autres Méthodes, dont la premiere donne les moyens de résoudre ces mêmes égalitez par la Geométrie et la seconde, pour resoudre plusieurs questions de Diophante qui n'ont pas encore esté resolues*, Paris 1691.

[3] In his *Traité d'algebre* (1690), Rolle gave the "method of cascades," for a cascade of an equation

$$f(x)=a_0+a_1x+\ldots+a_{n-1}x^{n-1}+a_nx^n=0$$

being understood the equation obtained by multiplying the terms of the original equation by the terms of the progression 0, 1, 2,...... n and dividing both sides by x; in other words the equation $f'(x)=0$. The theorem on which the method of cascades is based is not Rolle's theorem but a corollary to it, viz., between two successive real roots of $f'(x)=0$ there cannot be more than one real root of $f(x)=0$.

real root lying between two successive real roots of $f(x)=0$. Statements [1] which amount to Rolle's theorem are found in Euler's *Institutiones Calculi differentiales* which appeared in 1755 and which was until the appearance of Lagrange's *Théorie des fonctions analytiques* in 1797 the only really standard work on the Differential Calculus. The theorem is given by Lagrange, and also by Paolo Ruffini in his *Teoria generale delle equazioni* which appeared in 1799. But, according to Professor Cajori,[2] "none of the eighteenth century writers whom we have cited calls Rolle's theorem by the name of its discoverer or directly attributes it to Rolle, or lays stress upon the theorem, or even allows it to stand out conspicuously as a theorem; it is stated only in passing." "This attitude continues with many writers of the first half of the nineteenth century." The name of Rolle is not mentioned anywhere in Cauchy's *Résumé des leçons sur les calcul infinitésimal* (1823) or his *Leçons sur le calcul différentiel* (1829); it is absent from the pages of De Morgan's *Differential and Integral Calculus*, which appeared in 1842; it is also absent from the pages of the two well-known books on Differential Calculus which appeared in England in the period 1850-1860, *viz.*, those by B. Price and I. Todhunter. "In 1860 Giusto Bellavitis, a man unusually well-versed in the literature of the theory of equations, ascribes Rolle's theorem to Rolle and calls it the 'teorema del Rolle.' In 1868 the theorem is given in the German edition of Serret's *Algèbre Supérieure* under the name of 'Rollescher Satz.' Since then the theorem came to be generally ascribed to Rolle and to be called by his name." "During the second half of the nineteenth century we see the strange spectacle of Rolle's theorem, which had been in the theory of equations a star of the seventh or eighth magnitude, become in the wider region of mathematical analysis a star of the first magnitude. Rolle's theorem is used now in proving the theorem of mean-value, from which radiate some of the wonderful illuminations which make the modern development of the calculus so admirably rigorous."

§ 2.

2. Although the mean-value theorem in its now familiar form

$$f(x+h)=f(x)+h\,f'(x+\theta h) \tag{1}$$

appears for the first time in Cauchy's *Résumé* (1823), it was given in

[1] Articles 295, 297 (pp. 502-503 of Euler's Works, Series I, Vol. X).

[2] "On Michel Rolle's book 'Méthode pour résoudre les egalitez' and the history of 'Rolle's theorem'" (*Bibliotheca Mathematica*, Series 3, Vol. XI, 1910-1911, pp. 300-318).

nearly the above form by Lagrange who first obtained the remainder in Taylor's expansion for $f(x+h)$ after n terms and then put $n=1$. Professor Pringsheim[1] recognizes the two inequalities

$$\left.\begin{array}{l}\phi(x+z)-\phi(x)-z\,\phi'(x+p)>0,\\ \phi(x+z)-\phi(x)-z\,\phi'(x+q)<0,\end{array}\right\}(0<z\leq h), \tag{2}$$

where $\phi'(x+p)$ and $\phi'(x+q)$ are respectively the minimum and maximum of $\phi'(x+z)$ for $0\leq z\leq h$, as equivalent to the mean-value theorem. These inequalities appear in Lagrange's *Théorie des fonctions analytiques* in a generalized form, and in the above form in his *Calcul des fonctions* (1806) and in Ampère's paper "Recherches sur quelques points de la théorie des fonctions dérivées" (*Journ. de l'école poly.*) of the same year. Cauchy gives these great prominence in his *Résumé* and deduces the form (1) only as a corollary. "Perhaps the first author to bring together the theorem of mean-value and Rolle's theorem was Ossian Bonnet whose derivation of the theorem of mean-value from Rolle's theorem is given by Serret" in his *Cours de calcul différentiel et intégral* (1868). "Serret does not, however, mention the name of Rolle in the proof." The conditions under which (1) holds appear in half a dozen different forms in the works of the writers who followed Cauchy. The present accepted condition for the validity of (1) is substantially that of Bonnet and was first given with care by Dini in his lectures[2] at Pisa during the years 1871-1872. Dini's formulation and proof were utilized by Prof. G. Peano[3] to criticize both Bonnet's proof, and Jordan's proof as given in the first edition of his *Cours d'Analyse*, Vol. 1, of 1882. The conditions for the validity of (1) were made less restrictive by Prof. W. H. Young and Dr. G. C. Young;[4] and still less restrictive by Dr. A. N. Singh.[5]

3. I proceed now to give a brief historical account of the investigations of θ which seems to have been first studied by Cauchy and his school;

[1] "Zur Geschichte des Taylorschen Satzes" (*Bibliotheca Mathematica*, Series 3, Vol. I, 1900, pp. 433-479, specially pp. 444-450).

[2] Published in 1878 in the shape of the book *Fondamenti per la teorica delle funzioni di variabili reali.*

[3] *Nouvelles Annales* for 1884, pp. 45-47, 153-155, 252-256, 475-482. Jordan's proof referred to is the one which appeared in his *Cours d'Analyse de l'École Polytechnique*, t. 1, 1882.

[4] "On derivates and the theorem of the mean" (*Quarterly Journal of Mathematics*, Vol. XL, 1908, pp. 1-26).

[5] "On the mean-value theorem of the Differential Calculus" (*Bulletin of the Calcutta Mathematical Society*, Vol. XIX, 1928, pp. 43-49).

probably the result,[1] that for a fixed x, $\theta(+0)=\frac{1}{2}$, originated from them. The first writer to attempt the expansion of θ in powers of h was Whitcom[2] who gave in 1880 the first six co-efficients, the next two co-efficients were given by Mr. Bhola Nath Pal[3] in 1928. But the first important study of the functional nature of θ we owe to Prof. R. Rothe. Others who have studied the nature of θ are T. Hayashi, L. Sokolowski, Szász and Ganesh Prasad. The works of these five authors[4] have appeared only in the last nine years.

§ 3.

4. The various important forms of the mean-value theorem, apart from the generalizations of the Youngs and Singh, may be stated as follows, beginning with Bonnet's form and first going back to Cauchy's and then going forward to Dini's and Thomae's:—

I. *Bonnet's form*[5] (1868): Let $f(x)$ be a function of x, which remains continuous for the values of x comprised between given limits, and which, for those values, has a determinate differential co-efficient. Then if x_0 and x_0+h denote two values of x comprised between the aforesaid limits

$$f(x_0+h) - f(x_0)=hf'(x_0+\theta h),$$

θ being a quantity comprised between 0 and 1.

II. *Bertrand's form*[6] (1864): The mean-value theorem is not explicitly given anywhere by Bertrand but Lagrange's remainder in Taylor's expansion is deduced by the use of what we now call Rolle's theorem;

[1] This is the opinion of R. Rothe (*Math. Zeit.*, Bd. 9, p. 307).

[2] "On the expansion of $\phi(x+h)$" (*American J. of Math.*, Vol. 3).

[3] "On the expansion of θ in the mean-value theorem of the Differential Calculus" (*Bulletin of the Cal. Math. Soc.*, Vol. 19, pp. 143-146).

[4] Rothe, *Math. Zeit.*, Bd. 9, 1921, pp. 300-325; *Tohoku Math. J.*, Vol. 29, pp. 145-157, 1928.

Hayashi, *Science Reports, Tohoku Imp. Univ.*, 1925.

Sokolowski, *Tohoku Math. J.*, Vol. 31, 1929, pp. 177-191. Szász, *Math. Zeit.*, Bd. 25, 1926.

Ganesh Prasad: "On the function θ in the mean-value theorem of the Differential Calculus" (*Bulletin of the Calcutta Mathematical Society*, Vol. XX, 1929, pp. 155-184); "On the nature of θ in the mean-value theorem of the Differential Calculus" (*Bull. A. M. S.* 1930); "On Rolle's function θ as multiple-valued function" (*Proceedings of the Benares Mathematical Society*, Vol. X, 1929, pp. 1-10).

[5] Serret: *Cours d'calcul différentiel et intégral*, t. 1, p. 17.

[6] *Traité de Calcul Différentiel*, pp. 281-283.

neither the name of Rolle nor the mean-value theorem has been mentioned by Bertrand. As a particular case of the remainder theorem, the mean-value theorem comes according to Bertrand to the following: If a function $\phi(x)$ together with $\phi'(x)$ is continuous in an interval (a, b), then

$$\phi(b)-\phi(a)=(b-a)\,\phi'(a+\theta\overline{b-a}),\ \theta<1.$$

III. *Price's form* [1] (1852): If x_n and x_0 are two definite values of x, x_n being greater than x_0, and x_n-x_0 being a finite quantity, and if $F(x)$ be a function of x, which, as also its first-derived function, is finite and continuous for all values of x between x_n and x_0, then

$$F(x_n)-F(x_0)=(x_n-x_0)\,F'\{x_0+\theta(x_n-x_0)\},$$

θ being some *proper positive fraction*.

IV. *Todhunter's form* [2] (1852): same as in Moigno's given below.

V. *De Morgan's form* [3] (1842): If $\phi(x)$ be a function which is finite and without singular values from $x=a$ to $x=a+h$ inclusive, and if the differential co-efficient be the same then

$$\phi(a+h)-\phi(a)=h\phi'(a+\theta h)$$

for some positive value of θ less than unity.

VI. *Moigno's form* [4] (1840): If $f(x)$ together with $f'(x)$ is continuous when x passes from x_0 to x_0+h then

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h),$$

θ being a number, comprised between 0 and 1. The Cauchy's generalized mean-value theorem is first given and the above is deduced from it.

VII. *Cauchy's form* [5] (1823, 1829): When the function $f(x)$ has a finite value for $x=x_0$, and remains continuous together with its differential co-efficient $f'(x)$, from $x=x_0$ up to $x=x_0+h$, there exists between the limits 0 and 1 a value of θ such that

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h).$$

[1] *A Treatise on the Differential Calculus*, p. 169.
[2] *Treatise on the Differential Calculus*, p. 78.
[3] *The Differential and Integral Calculus*, p. 67.
[4] *Leçons de Calcul Différentiel*, p. 39.
[5] *Leçons sur le Calcul différentiel* (*Oeuvres*, S. 2, t. IV, p. 312).

5. VIII. *Hermite's form*[1] (1873): The mean-value theorem is not explicitly given but Rolle's theorem with the name of Rolle is given. Also the Lagrange's remainder is deduced. So, according to Hermite the mean-value theorem comes to the following: If $f(x)$ together with its differential co-efficient is continuous in an interval (x_0, X), then

$$f(X)-f(x_0)=(X-x_0)f'\{x_0+\theta(X-x_0)\}, \ \theta<1.$$

IX. *Thomae's form*[2] (1875): If $f(x)$ is finite and continuous in an interval (a, b) and has in the interval a continuous progressive differential co-efficient $f'(x+0)$, then

$$f(x_0+h)-f(x_0)=hf'(\xi+0),$$

ξ being between x_0 and x_0+h.

X. *Houel's form*[3] (1878): Same as Cauchy's (with this difference that θ is said to be subject to the inequality, $0\leq\theta\leq1$).

XI. *Dini's form*[4] (1878): If $f(x)$ is finite and continuous in the whole of an interval, and, excepting at the utmost the ends of this interval, in all the other points has always a differential co-efficient which is finite and determinate or which, being infinite, is determinate in sign; then, denoting by x and $x+h$ (h positive or negative) two points whatever of this interval (the ends included) and by θ a number comprised between 0 and 1 (0 and 1 excluded) which depends on x and on h, we have always

$$f(x+h)=f(x)+hf'(x+\theta h).$$

§ 4.

6. Before Prof. W. H. Young and Dr. G. C. Young published[5] their generalization of the mean-value theorem it was believed that the absence of the differential co-efficient at any point inside the interval $(x, x+h)$ might render invalid the relation

$$f(x+h)=f(x)+hf'(x+\theta h), \ 0<\theta<1.$$

This point has been carefully elaborated in Schlömilch's *Höhere Analysis*, Bd. 1, 1881. The first investigators to relax the conditions for the

[1] *Cours d'Analyse de l'École Polytechnique*, pp. 48-50.

[2] *Einleitung in die Theorie der bestimmten Integrale*, p. 10. Note that the condition of the continuity of $f'(x+0)$ carries with it as a consequence the existence and continuity of $f(x)$.

[3] *Cours de Calcul Infinitésimal*, p. 146.

[4] *Fondamenti per la teorica delle funzioni di variabili reali*, p. 75.

[5] *L.c.*

existence of the differential co-efficient were the two Youngs. Their generalization is the following:

XII. If there is no distinction of right and left with regard to the derivates of $f(x)$, then there is a point in the completely open interval (a, b) at which $f(x)$ has a differential co-efficient, and the value of that differential coefficient is precisely

$$m(a, b) \equiv \frac{f(b)-f(a)}{b-a},$$

$$i.e., \frac{f(b)-f(a)}{b-a}=f'\{a+\theta(b-a)\}, \ 0<\theta<1.$$

The special case in which $f(b)=f(a)=0$ is called the generalized Rolle's theorem, and the following is called *Rolle's theorem for derivates*: If $f(x)$ is a finite function or has infinities only at a nowhere dense set[1], and is further continuous throughout the closed interval (a, b) and is zero at the end points a, b; then there is a point ξ of the completely open interval (a, b) at which one of the upper derivates is not positive and the other lower derivate is not negative,

$$i.e., \ f^{+}(\xi) \leq 0 \leq f_{-}(\xi),$$

or the alternative inequality, interchanging left and right,

$$f^{-}(\xi) \leq 0 \leq f_{+}(\xi).$$

7. Singh[1] has gone further than the Youngs and his generalization still more relaxes the conditions imposed on the derivates; Singh's generalization, is the following:

XIII. If $f(x)$ be a continuous function defined in the closed interval (a, b), such that

(1) there is no point within (a, b) at which one of the derivatives, (*i.e.*, progressive differential co-efficient or regressive differential co-efficient) exists while the other does not exist, and (2) at each point within (a, b) the upper and lower derivates on one side lie within or are equal to the upper and lower derivates on the other side; then there exists a point in the completely open interval (a, b) at which the differential co-efficient exists and its value is equal to

$$\frac{f(b)-f(a)}{b-a}.$$

Singh calls the following theorem, *Rolle's theorem for derivatives*: If $f(x)$ be a continuous function defined in the interval (a, b), such that the upper

[1] L.c.

and lower derivates on one side lie within or are equal to the upper and lower derivates on the other; then there is a point in the completely open interval (a, b) such that at least one of the two derivatives exists at this point and its value is

$$\frac{f(b)-f(a)}{b-a}.$$

§ 5.

8. I proceed to give you an idea of what our present knowledge of the functional nature of θ as a *single-valued function of h* is.

(*a*) In 1880, Whitcom[1] published his investigation of the expansion of θ in powers of h, postulating implicitly not only the expansibility of $f(x+h)$ in Taylor's series in powers of h but also the expansibility of $\theta(h)$ in powers of h. He gave the *general* system of n simultaneous equations for finding the n co-efficients $A_1, A_2, \ldots A_n$ in the expansion

$$\theta = \tfrac{1}{2} + hA_1 + h^2A_2 + \ldots h^nA_n + \ldots \text{to infinity.}$$

The actual values of A_1, A_2, A_3, A_4 and A_5 have been calculated by Whitcom, but the general expression for A_n has not been given.

(*b*) B. N. Pal[2] made an attempt to find the general expression for A_n but succeeded only in determining two more co-efficients, *viz.*, A_6 and A_7. Both Rothe and Hayashi seem to have been ignorant of Whitcom's result, as their investigations to determine A_1 and A_2 show.

9. The first attempt to investigate, as fully as the present state of analysis will allow, the functional nature of the number θ in the mean-value theorem has been made recently by Ganesh Prasad. For the sake of simplicity he takes x to be 0 and $f(0)$ to be 0. In a paper,[3] read before the Calcutta Mathematical Society in July, 1929, he has first given a number of fundamental theorems, *viz.*, (1) If θ is a single-valued function of h, then it is necessarily continuous everywhere with the possible exception of $h=0$. (2) If $f'(h)$ is monotone and continuous in the domain $(0, 1)$ of h, then there is one-to-one correspondence between h and $\xi \equiv \theta h$, h varying in the domain $(0, 1)$ and ξ in its own domain, say $\triangle$. (3) If $\theta(h)$ is single-valued and continuous, $\theta'(h)$ need not exist for every value of h. (4) If θ is single-valued, the function $f'(t)$ must not have an infinite number of maxima and minima in the neighbourhood of any point t in its domain $\triangle$. Prasad next considers the conditions for the existence of $\theta(+0)$ or $\theta'(+0)$. He further discusses certain types of θ each of which is non-differentiable at the points of an everywhere dense set. Finally, he

[1] *L.c.* [2] *L.c.*

[3] *Bulletin of the Calcutta Mathematical Society*, Vol. 20, pp. 155-184.

considers the question: What condition must be satisfied by a prescribed function $\theta(h)$ in order that there should exist a corresponding function $f(h)$?

§ 6.

10. The fact that θ may be a *multiple-valued function of h* was recognised by Cambridge mathematicians [1] as early as 1891, and that fact was later discussed at some length by E. R. Hedrick.[2] Very recently G. Prasad [3] has studied the function θ, which he calls Rolle's function, as a multiple-valued function. He has introduced the notion of the principal value θ_1 of the function, defining it as the greatest of the multiple-values of θ for a given h, and proved that (*i*) $\theta_1(h)$ may be discontinuous, (*ii*) that, if continuous, it may not be everywhere differentiable and (*iii*) that, in the case in which $f'(h)$ is nowhere differentiable, $\theta_1(h)$ is also nowhere differentiable. Arranging the various values of θ in order of magnitude in an enumerable sequence $\{\theta_n\}$, what holds for θ_1 may be shown to hold for a given member $\theta_n(h)$ of the sequence.

11. R. Rothe's paper of 1921 also treats θ as a function of the two variables x and h, and proves the following theorems relating to it:

(*i*) Let x and h $(\neq 0)$ be real, $a \leq x \leq b$ and also $a \leq x+h \leq b$; further let $f(x)$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$; then for the interval (a, b) and for every sub-interval of it the mean-value theorem holds. If θ is independent of x and h, then it must be $\frac{1}{2}$ and $f(x)$ must be of the form $\alpha x^2+\beta x+\gamma$ with $\alpha \neq 0$.

(*ii*) With the suppositions made in (*i*) about $f(x)$, if $\theta(h)$ is independent of x and dependent only on h, then of all those functions $\theta(h)$ which are *single-valued and differentiable*, the only one which satisfies the mean-value theorem is

$$\theta = \frac{1}{ah} \log \frac{e^{ah}-1}{ah}$$

and the corresponding $f(x)$ is $\alpha e^{ax}+\beta x+\gamma$ with $a \neq 0$.

[1] In an examination paper of June 10, 1891, for the Jesus, Christ's, Magdalene, Emmanuel and Sidney Sussex colleges the following question appears: "Point out when θ has more than one value corresponding to a given value of x and h."

[2] "On a function which occurs in the law of the mean" (*Annals of Mathematics*, Vol. 7, 1906, pp. 177-192). In the discussion of some properties of $\xi = h\theta$, an element of confusion is the absence of the explicit isolation of each multiple value of ξ and its study as a single-valued function of h.

[3] "On Rolle's function θ as multiple-valued function" (*Proceedings of the Benares Mathematical Society*, Vol. X, 1929). See also his paper in Vol. XI of the same journal, "On the zeroes of Weierstrass's non-differentiable function," and a paper "On Rolle's function θ in the mean-value theorem for the case of a nowhere differentiable $f'(x)$" which will soon appear in the *Bulletin of the Calcutta Mathematical Society*.

(*iii*) With the suppositions made in (*i*) about $f(x)$, if $\theta\,(x, h)$ is independent of h and depends only on x, then no such variable function θ exists which also possesses

$$\frac{d\theta}{dx}.$$

Rothe also considers in the same paper the following question about θ: what conditions must be satisfied by θ as a function of x and h in order that there should be a corresponding function $f(x)$ to satisfy the mean-value theorem

$$f(x+h)=f(x)+h\,f'(x+\theta h)\,?$$

In treating this question Rothe has postulated the continuity and existence of

$$\theta_1\equiv\frac{\partial\theta}{\partial x_1},\quad \theta_2\equiv\frac{\partial\theta}{\partial x_2},\quad \theta_{12}\equiv\frac{\partial\theta_1}{\partial x_2}\equiv\frac{\partial\theta_2}{\partial x_1},\quad \theta_{121}\equiv\frac{\partial\theta_{12}}{\partial x_1},\quad \theta_{122}\equiv\frac{\partial\theta_{12}}{\partial x_2},$$

where $x_1=x$ and $x_2=x+h$.

§ 7.

12. Before I conclude this historical introduction, I should like to mention briefly a number of miscellaneous theorems and results each of which is, in some sense, connected with the mean-value theorem or the Rolle's function θ as a kind of generalization. I will begin with the mention of Cauchy's generalized mean-value theorem

$$\frac{\phi(x+h)-\phi(x)}{\mathrm{F}(x+h)-\mathrm{F}(x)}=\frac{\phi'(x+\theta_1 h)}{\mathrm{F}'(x+\theta_1 h)},\quad 0<\theta_1<1.$$

In addition to the well-known generalizations [1] of the above theorem by Genocchi and Peano, may be mentioned the theorems of similar

[1] The better-known of these is

$$\begin{vmatrix} f(x_0+h) & \phi(x_0+h) & \mathrm{F}(x_0+h) \\ f(x_0) & \phi(x_0) & \mathrm{F}(x_0) \\ f'(x_0+\theta h) & \phi'(x_0+\theta h) & \mathrm{F}'(x_0+\theta h) \end{vmatrix}=0,\ 0<\theta<1.$$

See *Calcolo differenziale e principii di calcolo integrale*, 1884, p. xiv. For $f(x)=1$, the above reduces to Cauchy's generalized mean-value theorem.

character given recently by D. Pompeiu,[1] T. Hayashi [2] and Takahashi.[3] Pompieu's theorem is, that under certain conditions for ϕ and F

$$\{\phi(x_0+h)-\phi(x_0)\}\{F(x_0+h)-F(x_0)\}=h^2\phi'(x+\theta_2 h)\;\; F'(x+\theta_2 h),\; 0<\theta_2<1.$$

The theorems of Hayashi and Takahashi are complicated as they involve integrals, although each is based on Rolle's theorem.

13. The next group of results to be mentioned are those relating to the number θ_n occurring in Lagrange's remainder $\frac{h_n}{n!} f^{(n)}(x+\theta_n h)$ and the number θ_1 in Cauchy's generalized mean-value theorem. As in the case of the ordinary θ of the mean-value theorem, Whitcom [4] gave the expansion of θ_n in powers of h with the co-efficients of h, h^2, h^3; the constant he showed to be $\frac{1}{n+1}$. The function θ_n has been studied by R. Rothe and the following results, among others, have been found by him:

(a)[5] In order that a function $f(x)$ satisfying Taylor's theorem with the remainder θ_n of the nth order be satisfied by a value of θ_n independent of x and h, it is necessary and sufficient that $f(x)$ be a polynomial of the $(n+1)$th degree, and that $\theta_n=\frac{1}{n+1}$.

(b)[6] $\lim_{h=+0} \theta_n=\frac{1}{n+1}$, if $f^{(n+1)}(x)$ is existent and continuous for $a<x<b$ and at the same time not identically zero.

(c)[7] $\theta'_n(+0)=\frac{1}{n+1}\left\{\frac{1}{n+2}-\frac{1}{2(n+1)}\right\}\cdot\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}$, under certain conditions. The work of Rothe in this connection is open to the same criticism as in the case of Rolle's function θ.

[1] "Sur une forme du theoreme des accroissements fini" (*Bulletin Scient. Acad. Roumania*, 1924-1925, pp. 61-64).

[2] *The Science Reports of the Tohoku Imp. Univ.*, Vol. 15, 1926, pp. 185-191.

[3] *Tohoku Math. Journal*, Vol. 30, 1929, p. 432.

[4] *L. c.*, pp. 349-351.

[5] *Math. Zeit.*, Bd. 9, pp. 308-309.

[6] *Math. Zeit.*, Bd. 9, pp. 309-310.

[7] *Tohoku Math. J.*, Vol. 29, 1928, p. 151.

The θ_1 occurring in Cauchy's generalized mean-value theorem has been studied by Takahashi after Rothe's method; and it has been shown, for example, that if θ_1 be independent of x and h it must be 1/2, and F and ϕ must come under one of three possible cases. Takahashi's study is open to the same criticism as Rothe's.

14. The last group of results relate to the case of two or more independent variables. Much is still left to be discussed in this connection; the only results worth mention here are the various expressions for

$$f(x+h,\ y+k)-f(x,\ y):$$

(*a*) The result that

$$f(x_0+h,\ y_0+k)-f(x_0,y_0)=k\psi(x_0+h,\ y_0+\theta_1 k)+h\phi(x_0+\theta h, y_0),$$

ϕ standing for $\dfrac{\partial f}{\partial x}$, ψ for $\dfrac{\partial f}{\partial y}$ and θ, θ_1 for numbers each lying between 0 and 1, is an immediate deduction from the mean-value theorem.

(*b*) The result that

$$f(x_0+h,\ y_0+k)-f(x_0,\ y_0)=h\ \phi(x_0+\theta h,\ y_0+\theta k)+k\psi(x_0+\theta h,\ \ y_0+\theta k),$$

$0<\theta<1$, is based on the mean-value theorem and an artifice by which x and y are looked upon as functions of only one variable, say t, so that

$$x=x_0+ht, \qquad y=y_0+kt.$$

(*c*) Goursat and Hedrick[1] have given a third result, *viz.*,

$$f(x+h,\ y+k)-f(x,\ y)=h\ f'_x\ (x+\theta h,\ y+k)+kf'_y\ (x,\ y+\theta k),$$

$$0\leq\theta\leq 1.$$

[1] *A Course in Mathematical Analysis*, Vol. I, p. 15.

SECOND LECTURE

Proofs of the Mean-value Theorem.

§ 8.

15. To-day's lecture will deal at some length with the various forms of the mean-value theorem as given by writers from Cauchy to Dini and with the proofs of those forms. In many cases the original proofs will be given and criticized if necessary. As in the first lecture I gave Ossian Bonnet's enunciation first, so I proceed to give to-day first his proof[1] almost word by word:

"Let $f(x)$ be a function of x which remains continuous for the values of x *comprised between the given limits*, and which, for these values, has a determinate differential co-efficient $f'(x)$. If x_0 and X denote two values of x *comprised between the aforesaid limits*, then one shall have

$$\frac{f(\mathrm{X})-f(x_0)}{\mathrm{X}-x_0}=f'(x_1),$$

x_1 being a value comprised between x_0 and X.

In fact, the ratio

$$\frac{f(\mathrm{X})-f(x_0)}{\mathrm{X}-x_0}$$

has, by hypothesis, a finite value; and, if one calls this value A, one shall have

$$\{f(\mathrm{X})-\mathrm{AX}\}-\{f(x_0)-\mathrm{A}x_0\}=0. \tag{1}$$

Let us denote by $\phi(x)$ the function of x defined by the formula

$$\phi(x)=\{f(x)-\mathrm{A}x\}-\{f(x_0)-\mathrm{A}x_0\}. \tag{2}$$

then one shall have, because of the equality (1),

$$\phi(x_0)=0,\ \phi(\mathrm{X})=0,$$

so that $\phi(x)$ vanishes for $x=x_0$ and for $x=\mathrm{X}$.

[1] Serret's *Cours de Calcul Differentiel*, 1868, pp. 16-19.

Let us suppose, for the fixity of ideas, $X > x_0$, and let us make x increase from x_0 to X; the function $\phi(x)$ is 0 to start with. If one admits that it is not constantly 0, for the values of x comprised between x_0 and X, *it is necessary that it commences to increase by taking positive values, or to decrease by taking negative values*, may be, starting from $x=x_0$ or from a value of x comprised between x_0 and X. If the values in question are positive, as $\phi(x)$ is continuous and as it must vanish for $x=X$, it is evident that there shall be a value x_1 between x_0 and X such that $\phi(x_1)$ shall be greater than, or at least equal to, the neighbouring values $\phi(x_1-h)$, $\phi(x_1+h)$, h being as small a quantity as we please. If the function on ceasing to be 0 takes negative values, the same reasoning proves that there exists a value x_1 between x_0 and X such that $\phi(x_1)$ shall be less than, or at the utmost equal to, the neighbouring values $\phi(x_1-h)$, $\phi(x_1+h)$.

Thus in both the cases, the value of x_1 shall be such that the differences

$$\phi(x_1-h)-\phi(x_1),\ \phi(x_1+h)-\phi(x_1)$$

will be of the same sign, and, consequently, the ratios

$$\frac{\phi(x_1-h)-\phi(x_1)}{-h},\ \frac{\phi(x_1+h)-\phi(x_1)}{h} \tag{3}$$

will be of opposite signs.

It is necessary to remark that we do not exclude the hypothesis in which one of the preceding ratios reduces to 0, the hypothesis which will require that $\phi(x)$ has the same value for the values of x in a finite interval. In particular, if the function $\phi(x)$ is constantly 0 for the values of x comprised between x_0 and X, the ratios (3) are both 0.

The ratios (3) tend towards the same limit as h tends to 0, for we admit that the function $f(x)$ has a determinate differential coefficient and the same consequently holds for $\phi(x)$; always these ratios have opposite signs: therefore their limit is 0. Thus one has

$$\lim \frac{\phi(x_1+h)-\phi(x_1)}{h}=0,$$

or, because of the equation (2),

$$\lim \left\{\frac{f(x_1+h)-f(x_1)}{h}-A\right\}=0,$$

i.e., $$A=\lim \frac{f(x_1+h)-f(x_1)}{h}=f'(x_1).$$

One has therefore

$$\frac{f(\mathrm{X})-f(x_0)}{\mathrm{X}-x_0}=f'(x_1), \qquad (4)$$

as was announced.

We have supposed that $\mathrm{X} > x_0$, but, as the preceding formula does not change with the interchange of the letters x_0, X, it is independent of this hypothesis.

If one puts

$$\mathrm{X}=x_0+h,$$

the quantity x_1, comprised between x_0 and x_0+h can be represented by $x_0+\theta h$, θ being a quantity comprised between 0 and 1; one may therefore write

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h)."$$

16. It is necessary to make a few remarks about the above proof:—

(a) Although the function $f(x)$ is explicitly assumed to be continuous only "for the values of x comprised between the given limits" x_0 and X and not at those limits themselves, the continuity at those limits is tacitly assumed; for, otherwise, the essential element in the proof, *viz.*, that $\phi(x)$ shall attain its upper bound (or lower bound) at a definite point x_1 *between* the given limits, will not necessarily be true.

(b) The proof does not assume the existence of $f'(x)$ at x_0 or X.

(c) The proof does not assume either the continuity of $f'(x)$ anywhere or its finiteness.

(d) The proof is *slightly defective*, as pointed out by G. Peano,[1] because of the statement: "If one admits that it is not constantly 0, for the values of x comprised between x_0 and X, it is necessary that it commences to increase by taking positive values, or to decrease by taking negative values, may be, starting from $x=x_0$ or from a value comprised between x_0 and X."

If we take $\phi(x)$ to be $(x-x_0)\sin\dfrac{1}{x-x_0}$, it is clear that, although $\phi(x_0)$ is 0, it cannot be said that $\phi(x)$ begins to increase or to decrease, because in any interval ever so small round x_0 the function makes infinite number of fluctuations.

[1] "Extrait d'une lettre de M. le Dr. J. Peano" (*Nouvelles Ann.*, 1884, pp. 45-47).

§ 9

17. As Bertrand did not consider the mean-value theorem except as a deduction from the remainder theorem of Lagrange, I take up now the consideration of Price's proof, which is obviously open to the objection that the equations in (1) below *are not rigorously valid*:—

"If x_n and x_0 are two definite values of x, x_n being greater than x_0, and $x_n - x_0$ being a finite quantity; and if $F(x)$ is a function of x, which, as also its first derived function, is finite and continuous for all values of x between x_n and x_0, then

$$F(x_n) - F(x_0) = (x_n - x_0)F'\{x_0 + \theta(x_n - x_0)\},$$

θ being some proper positive fraction.

Let the difference $x_n - x_0$ be divided into n parts, and let $x_1, x_2, \ldots x_{n-1}$ be the values of x corresponding to the $n-1$ points of division; and let us, moreover, suppose n to be so large that each of the divided elements, $x_1 - x_0, x_2 - x_1, \ldots x_n - x_{n-1}$, is an infinitesimal.

Then, observing the definition of a derived function, we have the following series of equations:

$$\left.\begin{aligned} F(x_1) - F(x_0) &= (x_1 - x_0)F'(x_0), \\ F(x_2) - F(x_1) &= (x_2 - x_1)F'(x_1), \\ \ldots \quad \ldots \quad \ldots &\quad \ldots \quad \ldots \quad \ldots \\ F(x_n) - F(x_{n-1}) &= (x_n - x_{n-1})F'(x_{n-1}); \end{aligned}\right\} \quad \ldots (1)$$

whence, adding all the first and second members of the series of equations, the sum of the first is $F(x_n) - F(x_0)$, and the sum of the second is the product of the sum of the first factors, *viz.*, $(x_n - x_0)$, and some mean value of the second factors, that is,

$$F(x_n) - F(x_0) = (x_n - x_0)F'\{x_0 + \theta(x_n - x_0)\},$$

in which θ is some positive proper fraction."

§ 10

18. In refreshing contrast to Price's proof, is the following careful proof given by Cauchy nearly 30 years earlier.[1]

[1] *Resumé* (see *Oeuvres*, S. 2., t. 4, pp. 44-45).

"If the function $f(x)$ being continuous between the limits $x=x_0$, $x=X$, one denotes by A the least and by B the greatest of the values of the differential co-efficient $f'(x)$ in the interval, then the ratio

$$\frac{f(X)-f(x_0)}{X-x_0} \quad \dots \quad (4)$$

of the finite differences is necessarily comprised between A and B.

Let us denote by δ, ϵ two very small numbers, the first being chosen in such a manner that for the numerical values of i less than δ and for any value whatever of x comprised between the limits x_0, X, the ratio

$$\frac{f(x+i)-f(x)}{i}$$

remains always greater than $f'(x)-\epsilon$ and less than $f'(x)+\epsilon$. If between the limits x_0, X, one interposes $(n-1)$ new values of the variable x, say

$$x_1, x_2, \dots x_{n-1}$$

so as to divide the difference $X-x_0$ into elements

$$x_1-x_0, x_2-x_1, \dots X-x_{n-1}$$

which being all of the same sign have numerical values less than δ, the fractions

$$\frac{f(x_1)-f(x_0)}{x_1-x_0}, \frac{f(x_2)-f(x_1)}{x_2-x_1} \dots \frac{f(X)-f(x_{n-1})}{X-x_{n-1}}, \quad \dots \quad (5)$$

being found comprised, the first between the limits $f'(x_0)-\epsilon$, $f'(x_0)+\epsilon$, the second between the limits $f'(x_1)-\epsilon$, $f'(x_1)+\epsilon$, ... will be all greater than the quantity $A-\epsilon$ and less than the quantity $B+\epsilon$. Moreover, these fractions (5) having denominators of the same sign, if one divides the sum of the numerators by the sum of the denominators one shall obtain a mean fraction, that is to say, comprised between the greatest and the least of those which one considers.

The expression (4) with which the mean coincides, shall be therefore enclosed between the limits $A-\epsilon$, $B+\epsilon$, and as this conclusion subsists however small be the number ϵ, one may affirm that the expression (4) shall be comprised between A and B.

Corollary. If the differential coefficient $f'(x)$ is itself continuous between the limits $x=x_0$, $x=X$, in passing from one limit to the other, this function shall vary in such a manner as to remain always comprised

between A and B and to take successively all the intermediate values. Therefore every mean quantity between A and B shall be a value of $f'(x)$ corresponding to a value of x between the limits x_0 and $X=x_0+h$ or, what comes to the same thing, a value of x of the form

$$x_0+\theta h=x_0+\theta\,(X-x_0),$$

θ denoting a number less than 1."

19. It is necessary to make the following remarks on the above proof:

(a) What Cauchy explicitly assumes about $f'(x)$ is *its existence and finiteness for every value of x in the closed interval* (x_0, X) ; in the corollary it is implied that *continuity* is *not* necessary for the proof that

$$\frac{f(X)-f(x_0)}{X-x_0}$$

lies between A and B.

(b) In reality the proof, by taking δ and ϵ to be two such quantities that "for any value whatever of x comprised between the limits x_0, X, the ratio $\frac{f(x+i)-f(x)}{i}$ remains always greater then $f'(x)-\epsilon$ and less than $f'(x)+\epsilon$" as long as i is less than δ, assumes implicitly that $\frac{f(x+i)-f(x)}{i}$ *uniformly* tends to $f'(x)$ for every value of x. *This is equivalent to the continuity of $f'(x)$.*

Take, *e.g.*, $x_0=0$, $X=1$, $f(x)=x^2 \sin 1/x$. Then $f'(x)$ is existent and finite for every value of x in $(0, 1)$, the ends included. But, because of the discontinuity of $f'(x)$ at $x=0$, $\frac{f(x+i)-f(x)}{i}$ does not uniformly tend to $f'(x)$ in any interval which has 0 as an end-point. For, taking $x=\frac{1}{(2m+1)\pi}$, $i=\frac{1}{2m\pi}-\frac{1}{(2m+1)\pi}$, where m is integral, $f(x+i)=0$, $f(x)=0$, $f'(x)=1$, so that the difference $\frac{f(x+i)-f(x)}{i}-f'(x)$, is numerically equal to 1 and is not a vanishing quantity with increasing m.

§ 11

20. Another fairly careful proof is De Morgan's and is reproduced below. "Let there be two limits a and $a+h$, such that neither for them

nor between them, are there any singular[1] values of $\phi(x)$. Thus, for log x, from $x=2$ to $x=3$, there is no singular value, nor is log 2 of log 3 either of them singular. We have now P, a *comminuent*[2] with $\triangle x$, whatever the value of x may be, between a and $a+h$. Consequently, P and $\triangle x$ will still remain comminuent, even though, while $\triangle x$ diminishes, x should vary in any manner between a and $a+h$. Thus, for instance $\triangle x$ and $x\ \triangle x$ are comminuents, even though while $\triangle x$ diminishes, x should vary in any manner between a and $a+h$.

Let us suppose $\triangle x$ to be the nth part of h, so that $\triangle x$ diminishes without limit as n increases without limit. Let P, which is a function of x and $\triangle x$, be denoted by $f\ (x,\ \triangle x)$, and we then have

$$\frac{\phi(x+\triangle x)-\phi(x)}{\triangle x}=\phi'(x)+f(x,\ \triangle x);$$

now substitute successively $x+\triangle x$ for x until we come to have $\phi(x+n\triangle x)$ or $\phi\ (x+h)$ in the numerator, which will give the following set of equations (n in number) :—

$$\frac{\phi(x+\triangle x)-\phi(x)}{\triangle x}=\phi'(x)+f(x,\ \triangle x);$$

$$\frac{\phi(x+2\triangle x)-\phi(x+\triangle x)}{\triangle x}=\phi'(x+\triangle x)+f(x+\triangle x,\ \triangle x),$$

$$\frac{\phi(x+3\triangle x)-\phi(x+2\triangle x)}{\triangle x}='(x\phi+2\triangle x)+f(x+2\triangle x,\ \triangle x),$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$\frac{\phi(x+\overline{n-1}\ \triangle x)-\phi(x+\overline{n-2}\ \triangle x)}{\triangle x}=\phi'(x+\overline{n-2}\triangle x)$$
$$+f(x+\overline{n-2}\ \triangle x,\ \triangle x)$$

$$\frac{\phi(x+n\ \triangle x)-\phi(x+\overline{n-1}\ \triangle x)}{\triangle x}=\phi'(x+n-1\quad \triangle x)$$
$$+f(x+\overline{n-1}\ \triangle x,\triangle x)$$

Form the fraction which has the sum of the numerators of the preceding for its numerator, and the sum of the denominators for its denominator, it being clear that all the denominators have the same sign.

[1] For the definition of the expression "singular value," see p. 44 of De Morgan's book. The definition is, however, not clear. Perhaps, all that can be safely assumed is that De Morgan meant $\phi(x)$ to have a singular value at a if $\phi(x)$, $\phi'(x)$ or $\phi''(x)$ had a discontinuity at a.

[2] P is said to be a comminuent if it tends to 0 as $\triangle x$ tends to zero, *e.g.*, $x\ \triangle x$,

This gives

$$\frac{\phi(x+\triangle x)-\phi(x)+\phi(x+2\triangle x)-\phi(x+\triangle x)+\ldots+\phi(x+n\triangle x)-\phi(\overline{x+n-1}\triangle x)}{n\triangle x}$$

$$\text{or } \frac{\phi(x+n\triangle x)-\phi(x)}{n\triangle x} \text{ or } \frac{\phi(x+h)-\phi(x)}{h},$$

which must therefore lie between the greatest and least of the preceding fractions, or of their equivalents, all contained under the formula

$$\phi'(x+k\triangle x)+f(x+k\triangle x, \triangle x).$$

Now let the first value of x be a, and let C and c be the values of x which give $\phi'(x)$ the greatest and least possible values it can have between $x=a$ and $x=a+h$. (*We have supposed that $\phi'(x)$ does not become infinite between these limits.*) And let C′ and K′ be the values of x and k which give $f(x+k\triangle x, \triangle x)$ the greatest value it can have between the limits, and c' and k' those which give it the least. Then still more do we know that

$$\frac{\phi(a+h)-\phi(a)}{h} \text{ lies between } \phi'(\mathrm{C})+f(\mathrm{C}'+K'\triangle x, \triangle x) \text{ and}$$

$$\phi'(c)+f(c'+k'\triangle x, \triangle x),$$

in which the two functions marked f are, as we have shown, comminuents with $\triangle x$. Now, if a quantity always lie between two others, it must lie between their limits......The limits of the preceding, when n increases or $\triangle x$ diminishes, are $\phi'(\mathrm{C})$ and $\phi'(c)$; whence we have the following theorem: If $\phi(x)$ be a function which is finite and without singular values from $x=a$ to $x=a+h$ inclusive, and if the differential coefficient be the same, and if C and c be the values of x which make $\phi'(x)$ greatest and least between these limits, then it follows that

$$\frac{\phi(a+h)-\phi(a)}{h} \text{ lies between } \phi'(\mathrm{C}) \text{ and } \phi'(c).$$

Corollary. Since, by the law of continuity[1] of value, a function does not pass from its greatest to its least value without passing through *every* intermediate value, and since $\frac{\phi(a+h)-\phi(a)}{h}$ is *an* intermediate value of

[1] See pp. 45 and 46 of De Morgan's book.

$\phi'(x)$ between $\phi'(C)$ and $\phi'(c)$, and since $a+\theta h$ where, θ lies between 0 and 1....

$$\frac{\phi(a+h)-\phi(a)}{h}=\phi'(a+\theta h)."$$

§ 12

21. The following proof due to Cauchy [1] is of interest because it may be considered to be *a precursor of Bonnet's* proof; incidentally the generalized mean-value theorem of Cauchy is established:—" Let $f(x)$ and $F(x)$ be two real functions *which vanish for* $x=x_0$ and which remain continuous between the limits $x=x_0$, $x=X$. Let us suppose that the differential coefficient $F'(x)$ does not change its sign between the limits in question. If one calls A the least and B the greatest of the values which the ratio

$$\frac{f'(x)}{F'(x)}$$

receives in this interval, then the fraction

$$\frac{f(x)}{F(x)}$$

shall also remain comprised between the two limits A and B.

Proof. Because one shall have, by hypothesis, for all the values of x enclosed between the limits x_0 and X,

$$\frac{f'(x)}{F'(x)}-A>0,\quad \frac{f'(x)}{F'(x)}-B<0$$

and because the differential coefficient $F'(x)$ does not change its sign between these limits, one may affirm that in this interval one of the products

$$F'(x)\left\{\frac{f'(x)}{F'(x)}-A\right\}=f'(x)-AF'(x),$$

$$F'(x)\left\{\frac{f'(x)}{F'(x)}-B\right\}=f'(x)-BF'(x)$$

[1] *Leçons sur le calcul différentiel* (*Oeuvres*, Series 2, t. IV, pp. 308-313). This proof was first given in the *Addition* to the *Résumé* (*Oeuvres*, Series 2, t. IV, pp. 243-247).

shall be always positive, the other negative. Moreover these products are respectively equal to the differential coefficients of the functions

$$f(x)-\mathrm{AF}(x),\ f(x)-\mathrm{BF}(x).$$

Therefore one of these functions shall be always increasing, the other always decreasing, from $x=x_0$ to $x=\mathrm{X}$. Therefore, because they vanish simultaneously for $x=x_0$, the values which they receive for $x=\mathrm{X}$, *viz.*,

$$f(\mathrm{X})-\mathrm{AF}(\mathrm{X}),\ f(\mathrm{X})-\mathrm{BF}(\mathrm{X})$$

shall be of contrary signs and one can say the same of the quotients which those values give when divided by $\mathrm{F}(\mathrm{X})$, that is to say, the differences

$$\frac{f(\mathrm{X})}{\mathrm{F}(\mathrm{X})}-\mathrm{A}\ ,\ \frac{f(\mathrm{X})}{\mathrm{F}(\mathrm{X})}-\mathrm{B}.$$

Therefore the fraction $\dfrac{f(\mathrm{X})}{\mathrm{F}(\mathrm{X})}$ shall be comprised between A and B.

Corollary. If the differential coefficients $f'(x)$, $\mathrm{F}'(x)$ are themselves continuous between the limits $x=x_0$, $x=\mathrm{X}$, then when one passes from one limit to the other the ratio $\dfrac{f'(x)}{\mathrm{F}'(x)}$ shall vary in such a manner as to remain always comprised between the two values A and B and to take in succession all the intermediate values. Therefore every mean quantity between A and B, for example, the fraction $\dfrac{f(\mathrm{X})}{\mathrm{F}(\mathrm{X})}$ shall be a value of the ratio $\dfrac{f'(x)}{\mathrm{F}'(x)}$ corresponding to a value ξ of x enclosed between the limits x_0, X; so that $\dfrac{f(\mathrm{X})}{\mathrm{F}(\mathrm{X})}=\dfrac{f'(\xi)}{\mathrm{F}'(\xi)}$." Putting $\mathrm{X}=x_0+h$,

$$\frac{f(x_0+h)}{\mathrm{F}(x_0+h)}=\frac{f'(x_0+\theta h)}{\mathrm{F}'(x_0+\theta h)},\ 0<\theta<1.$$

Taking $f(x)-f(x_0)$ *for* $f(x)$ *and* $\mathrm{F}(x)=x-x_0$, *we have the mean-value theorem.*

§ 13.

22. Before giving Dini's proof, I proceed to say a few words about each of the following proofs: Todhunter's, Moigno's, Houel's.

(*a*) Todhunter's proofs are substantially the same as the first and second proofs of Cauchy and are free from objection because the continuity of the differential coefficients of the functions concerned is *explicitly* assumed. In the case of each proof, the generalized mean-value theorem is first proved.

(*b*) Moigno's proof is substantially the same as the second proof given in the preceding article.

(*c*) Houel's proof, although (at places) carelessly worded is substantially the same as Cauchy's first proof given in Art. 18.

23. The following is Dini's proof [1] of his form of the mean-value theorem:

"We form the function

$$\psi(x)=f(x)-f(a)-\frac{x-a}{b-a}\{f(b)-f(a)\}$$

which vanishes for $x=a$ and $x=b$.

This function $\psi(x)$ shall be evidently finite and continuous like $f(x)$ in the whole of the interval (a, b), and its differential coefficient, which we shall denote by $\psi'(x)$, shall be always finite and determinate or infinite and determinate in sign together with the differential coefficient of $f(x)$.

Now, as this function $\psi(x)$ is zero for $x=a$ and $x=b$ and is always continuous, if it will not be zero in the whole of the interval (a, b) there shall exist necessarily at least one determinate point x' in the *inside* of the aforesaid interval in which $\psi(x)$ shall actually attain its maximum value or minimum value; whence, unless $\psi(x)$ is always zero from a to b which cannot be the case, there shall always exist at least one determinate point x' in the inside of the interval for which there will be obtainable a number ϵ different from zero and positive such that for all the values of δ positive and less than ϵ the differences

$$\psi(x'+\delta)-\psi(x'),\ \psi(x'-\delta)-\psi(x'),$$

when not equal to 0, shall always have one and the same determinate sign for all the values of δ; and whence the ratios

$$\frac{\psi(x'+\delta)-\psi(x')}{\delta},\ \frac{\psi(x'-\delta)-\psi(x')}{-\delta}$$

when not zero, shall have for all the values of δ opposite signs, so that their limits for $\delta=0$, if existent, ought to be 0 or of opposite signs. But

[1] "*Fondamenti*," pp. 69-71.

these limits exist and ought to have one and the *same* value $\psi'(x')$, finite and determinate, or infinite and determinate in sign; because, on account of the hypothesis made, $f(x)$ and therefore $\psi(x)$ admits at any point *inside* (a, b) and consequently in x' a differential coefficient which is finite and determinate or infinite and determinate in sign......Hence

$$Lim \frac{\psi(x' \pm \delta)-\psi(x')}{\pm \delta}=0$$

is the only relation that can hold, and thus

$$f'(x')=\frac{f(b)-f(a)}{b-a} . \cdot\cdot$$

Putting

$$b-a=h, \ a=x, \ b \text{ is } x+h,$$

and we have

$$f(x+h)=f(x)+hf'(x+\theta h), \ 0<\theta<1.$$

§ 14.

24. (*a*) About the enunciations of the mean-value theorem, it may be safely stated that the essence of an enunciation is not that for two *given* values a, b of x, the conditions are required in order that

$$\frac{f(b)-f(a)}{b-a}=f'(a+\theta \, \overline{b-a}), \ 0 \angle \theta \angle 1;$$

such conditions can be given in infinite variety. For example, even if, at some intermediate points in (a, b), $f(x)$ were infinite or discontinuous or non-differentiable, it is easily seen, say by a graph, that a point between a and b could be chosen and the tangent to the curve $y=f(x)$ at that point made to be parallel to the chord joining the extreme points on the curve. The essential feature of the enunciation is, therefore, wanting in the various enunciations *excepting only those of Bonnet and Dini*, in each of which it is explicitly stated that the conditions required are such that for *every* sub-interval $(x, x+h)$ in (a, b),

$$\frac{f(x+h)-f(x)}{h}=f'(x+\theta h), \ 0<\theta<1.$$

(*b*) As regards the proofs, it may be correctly said that, excepting those of Bonnet and Dini both of which are nearly identical, all the

other proofs have *been borrowed from Cauchy and are identical with, or slight modifications of, one or the other of the two proofs given by Cauchy.* The *first proof* of Cauchy is vitiated by the mistake pointed out in (*b*) of Art. 19, the same mistake is to be found in Price's proof (which is otherwise carelessly worded), in Moigno's proof and in Houel's proof; it is strange that the same mistake appeared *in 1882 in the first volume of Jordan's Cours de Analyse.* The proofs given by Bertrand, Todhunter and Hermite escape serious criticism but unlike Cauchy's proof they assume at the outset the *continuity* of $f'(x)$. The second proof of Cauchy is open to the objection that it is taken to be almost self-evident that if a function has a differential coefficient $\left.\begin{matrix}>0\\<0\end{matrix}\right\}$ in an interval it is $\left.\begin{matrix}\text{increasing}\\\text{decreasing}\end{matrix}\right\}$ in that interval. De Morgan's proof is based on the assumption that nowhere has $f(x)$ or $f'(x)$ a *singular* value, the term " singular " being not *clearly* defined. Assuming that De Morgan means $f(x)$ to be finite and continuous with $f'(x)$, his proof is careful and is open to no objection.

(*c*) The *important question*: What is the necessary *and* sufficient condition in order that $f(x)$ being defined for an interval (a, b), the ends being included, for every sub-interval $(x, x+h)$ of (a, b) the relation

$$f(x+h)=f(x)+hf'(x+\theta h),\ 0<\theta<1,$$

holds? has not been answered as yet.

That, for the validity of the above relation, the finiteness as well as the continuity of $f(x)$ is necessary is obvious. But it is not necessary that $f'(x)$ be existent at every point inside (a, b); this is proved by the generalizations given by Prof. W. H. Young and Dr. G. C. Young, and by Dr. A. N. Singh.

Again, if, at a point α inside (a, b), $f'(\alpha+0)$ and $f'(\alpha-0)$ exist and are *unequal* then a sub-interval $(\alpha-h_1)$, $(\alpha+h_2)$ can be found for which the relation will not hold provided that $f(x)$ has *not* an infinite number of maxima and minima in the neighbourhood of α.

Some interesting results in connection with the question formulated in the beginning of this sub-para have been given by Prof. Brouwer.[1]

[1] L. E. J. Brouwer " Over differentiequotienten en differentiaalquotienten" (*Verslag Akad. Amsterdam*, Vol. 17, 1909, pp. 38-45; *Proceedings Acad. Amsterdam*, Vol. 11, 1909, pp. 59-66).

§ 15.

25. In 1906, D. Pompeiu[1] gave a proof of the mean-value theorem which differs from all the other proofs in this respect, that it is *not* based on Rolle's theorem which, in every one of the other proofs, is *first* established, and the mean-value theorem is *then* deduced from Rolle's theorem. I proceed now to give this proof of Pompeiu.

Let a function $f(x)$ be continuous in an interval (the ends included) and admit of a differential coefficient at every point inside that interval (the ends being excluded). Further, let a and b be two points in the given interval (the ends being included). Then form the ratio

$$R(a,\ b)=\frac{f(a)-f(b)}{a-b}.$$

The question is, to prove that there is at least one point between a and b the differential coefficient at which has precisely the value $R(a,\ b)$.

Now let

$$a_1=\tfrac{1}{2}(a+b).$$

Then we have

$$\frac{f(a)-f(b)}{a-b}=\tfrac{1}{2}\left\{\frac{f(a)-f(a_1)}{a-a_1}+\frac{f(a_1)-f(b)}{a_1-b}\right\},$$

or, briefly,

$$R(a,\ b)=\tfrac{1}{2}\{R(a,\ a_1)+R(a_1, b)\}. \qquad \ldots \quad (1)$$

Two cases arise:—(1) the two ratios in the right side of the above equality are equal in which case we may consider any one of them; (2) the two ratios are not equal so that one of them is greater, and the other less, than $R(a,\ b)$.

(a) Taking up the second case first, consider the function defined by

$$R(x,\ a_1)=\frac{f(x)-f(a_1)}{x-a_1}\ ,\ x\neq a_1,$$

$$R(x,\ a_1)=f'(x)\equiv\phi(x),\ x=a_1.$$

Then, obviously, for every value of x in the interval $(a,\ b)$ other than a_1, $R(x,\ a_1)$ is continuous in x; also because of $R(a_1, a_1)$ being defined to be $f'(a_1)$, it is continuous at a_1. Thus $R(x,\ a_1)$ is continuous for every

[1] "Sur le théorème des accroissements finis" (*Annales Scientifiques de l' Université de Jassy, 1906*).

value of x in (a, b), the ends being included; further, because of the inequality of $R(a, a_1)$ and $R(a_1, b)$, and (1), $R(a, b)$ lies between $R(a, a_1)$ and $R(a_1, b) = R(b, a_1)$. Thus the continuous function $R(x, a_1)$ must take the value $R(a, b)$ for some value b_1 of x intermediate between a and b, so that

$$R(b_1, a_1), \text{ i.e., } R(a_1, b_1), = R(a, b). \quad \dots \quad (2)$$

If the point b_1 coincides with a_1, then the theorem is proved as we will have

$$f'(a_1) = R(a, b).$$

In any case

$$| a_1 - b_1 | \leq \tfrac{1}{2} | a - b | .$$

If a_1 and b_1 are unequal, take

$$a_2 = \tfrac{1}{2}(a_1 + b_1).$$

Then, proceeding as in the case of $R(a, b)$, it is proved that between a_1 and b_1 a point b_2 exists such that

$$R(a_2, b_2) = R(a_1, b_1), \quad (3)$$

where

$$| a_2 - b_2 | \leq \tfrac{1}{2} | a_1 - b_1 | .$$

We continue this reasoning indefinitely if none of the points b_k coincides with the corresponding point a_k.

We obtain thus a set of intervals $\{I_k\}$ possessing the following properties:—

(1) the interval I_k, i.e., (a_k, b_k), is *inside* the interval I_{k-1} and equal in length at the utmost to the half of the length of I_{k-1} so that the length of I_k, i.e., $| a_k - b_k | \leq \frac{1}{2^k} | a - b |$;

(2) $R(a_k, b_k) = R(a, b)$, for every value of the integer k.

Therefore I_k tends to a limiting point c as k is indefinitely increased; but when a_k and b_k tend to c, from the definition of R it is obvious that $R(a_k, b_k)$ tends to $f'(c)$.

Thus it is proved that

$$\frac{f(a) - f(b)}{a - b} = f'(c),$$

where

$$a < c < b.$$

(β) If, in (1), $R(a, a_1)$ and $R(a_1, b)$ are equal, then, instead of (2) above, we have

$$R(a_1, b) = R(a, b)$$

so that $b_1 = b$ with

$$| a_1 - b_1 | = \tfrac{1}{2} (a - b).$$

Proceeding as under (a), we get the same result in the end as there. Thus the mean-value theorem is proved.

Criticism of Pompeiu's proof.

In order that Pompeiu's proof be valid, *something in addition to the existence of $f'(x)$* must be postulated. For, if $\{x_k\}$ and $\{y_k\}$ have the same limit c, it does not necessarily follow that $\dfrac{f(x_k)-f(y_k)}{x_k-y_k}$ tends to $f'(c)$. Take, *e.g.*, $a=0$, $b=1$, $f(x)=(x-\frac{1}{2})^2 \sin \dfrac{1}{x-\frac{1}{2}}$, $x_k=\frac{1}{2}+\dfrac{1}{2^k\pi}$,

$$y_k=\tfrac{1}{2}+\frac{1}{2^k\pi-\dfrac{\pi}{2}}.$$

Then x_k, y_k both tend to $\frac{1}{2}$; but $\dfrac{f(x_k)-f(y_k)}{x_k-y_k}$ does not tend to $f'(\frac{1}{2})$, *i.e.*, 0, but to $-\dfrac{\pi}{2}$.

§ 16.

26. I proceed now to consider the geometrical interpretations of the mean-value theorem. If we *assume* that $f(x)$ admits of a graph for the whole interval (a, b), an assumption which involves departure from regularity only at a *finite* number of points, then the *simplest geometrical interpretation* of the mean-value theorem is this: To the chord joining any two points $\{x, f(x)\}$, $\{x+h, f(x+h)\}$, say P, Q, there exists at least one tangent at an intermediate point p which is parallel to the chord. The truth of this interpretation is illustrated by the figures 1 and 2; the infinitude of $f'(x)$ at a point in the second figure not invalidating the theorem.

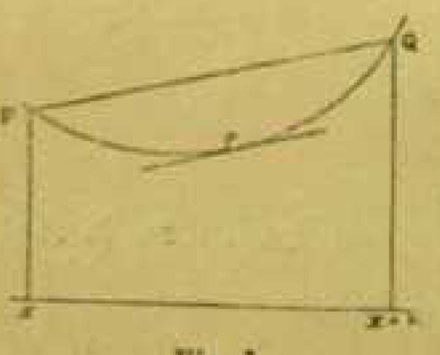

Fig. 1.

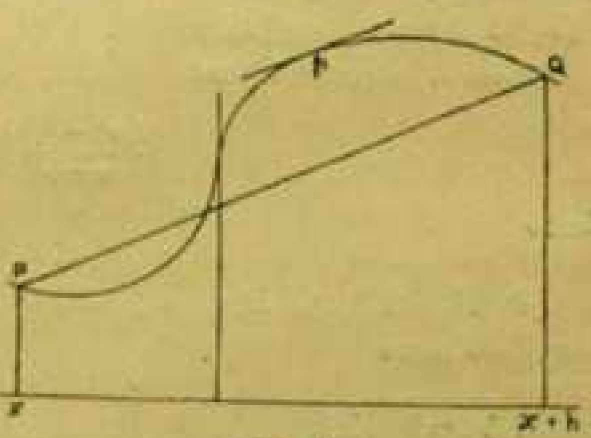

Fig. 2.

That the theorem *may* be invalid is illustrated by the figures 3, 4, 5.

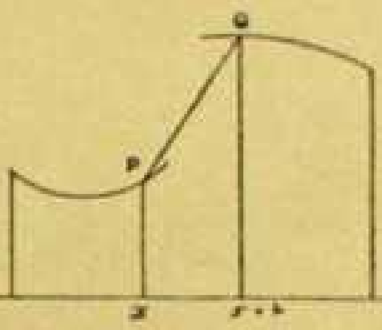

Fig. 3.

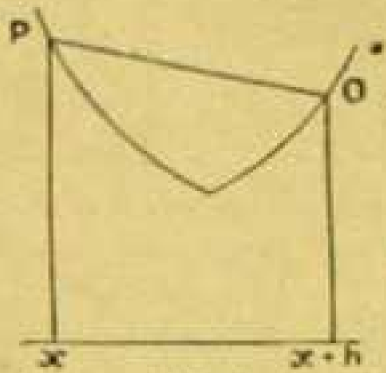

Fig. 4.

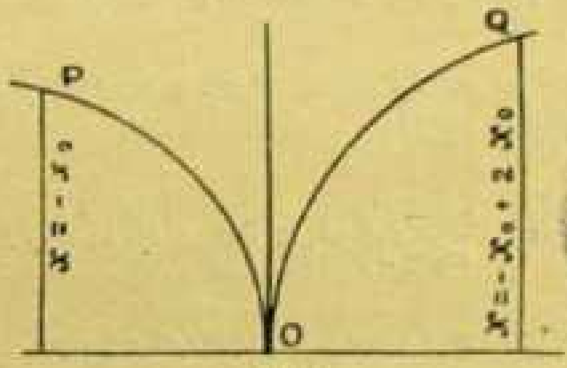

Fig. 5.

In Fig. 3. there is a discontinuity in $f(x)$ between P and Q; in Fig. 4. $f'(x)$ is non-existent at a point between P and Q; in Fig. 5 at O between P and Q there is a cusp but $f'(x)$ is non-existent there.

Other geometrical interpretations may be obtained by treating $f(x)$ as the area of the curve bounded by $y=f'(x)$ and the x-axis, or by treating $f(x)$ as a volume.

§ 17.

27. I will conclude this lecture *by deducing a number of important theorems* from the mean-value theorem in Dini's form, *i.e.*, with the assumptions that $f(x)$ is finite and continuous in the interval (a, b), the ends being included, and that $f'(x)$ is existent at every point in the inside of the interval, the ends being excluded.

(*a*) *Theorem A.*[1] If $f'(x)$ is zero at every point inside (a, b), then $f(x)$ is constant in the whole of the interval.

[1] Although it is self-evident that $f'(x)$ is zero in an interval if $f(x)$ is constant or that $f'(x)>0$ in an interval if at every point in that interval $f(x)$ is increasing, the converse propositions are *not* self-evident. The simplest logical proofs of those converse propositions are, as given here by applying the mean-value theorem.

Proof. Let x_1, x_2 be *any* two points in (a, b), then by the mean-value theorem

$$f(x_2)-f(x_1)=(x_2-x_1)\cdot f'(\xi), \qquad \dots \quad (1)$$

where ξ is a point between x_1 and x_2. But $f'(\xi)=0$ by the hypothesis. Therefore $f(x_2)=f(x_1)$; thus the theorem is proved.

(*b*) *Theorem B.*[1] If $f'(x)>0$ at every point inside (a, b), then $f(x)$ is a constantly increasing function; if $f'(x)<0$ at every point, then $f(x)$ is a constantly decreasing function.

Proof. Applying (1), since $f'(\xi)>0$, and $x_2>x_1$, $f(x_2)-f(x_1)>0$. Thus $f(x_2)>f(x_1)$, and it is proved that $f(x)$ constantly increases with x. Similarly, if $f'(x)<0$ at every point, it follows that $f(x)$ constantly decreases with increasing x.

(*c*) *Theorem C.* At *every* point of an interval, $f'(x)$ cannot be infinite.

Proof. Again applying (1),

$$f'(\xi)=\frac{f(x_2)-f(x_1)}{x_2-x_1}, \text{ a finite quantity.}$$

Therefore inside (x_1, x_2) there is at least one point ξ where $f'(x)$ is not infinite. Thus the theorem is proved.

(*d*) *Theorem D.* Unless $f(x)$ is constant throughout an interval, $f'(x)$ cannot be 0 at every point of that interval.

Proof. For, applying (1), $f'(\xi)$ is different from 0, and this theorem is proved.

(*e*) *Theorem E.* If at a point α in (a, b) *the limit of $f'(x)$ exists*, then that limit will be the value of $f'(\alpha)$, $f'(\alpha+0)$ or $f'(\alpha-0)$ according as α is inside (a, b), is a or b.

Proof. Let α be an interior point, then, by the mean-value theorem,

$$\frac{f(\alpha+h)-f(\alpha)}{h}=f'(\alpha+\theta h),\ 0<\theta<1.$$

Therefore, as h tends to 0, $\alpha+\theta h$ tends to α; consequently $f'(\alpha+\theta h)$ tends to a definite limit, say L, by the hypothesis. But by the definition of the differential coefficient at α,

$$\lim_{h=0} \frac{f(\alpha+h)-f(\alpha)}{h}=f'(\alpha).$$

Thus it is proved that $f'(\alpha)=\mathrm{L}$. Similarly, the other cases can be dealt with.

[1] *Ibid.*

(*f*) *Theorem F*. If, in addition to the assumptions in Dini's form of the mean-value theorem, $f'(x)$ *is continuous in* (a, b); then, corresponding to an arbitrarily small quantity $\delta > 0$, it is always possible to find a second quantity $\epsilon > 0$ and independent of x, such that

$$\left| \frac{f(x+h)-f(x)}{h} - f'(x) \right| < \delta, \qquad \dots \quad (2)$$

if $| h | < \epsilon$, whatever be the value of x in (a, b).

Proof. By the mean-value theorem, the left-side of (2) is equal to $| f'(x+\theta h) - f'(x) |$. But, by the hypothesis, $f'(x)$ is continuous in (a, b) and therefore uniformly continuous. Therefore, corresponding to δ, a quantity ϵ exists such that

$$| f'(x_2) - f'(x_1) | < \delta,$$

whatever values x_2 and x_1 may have in (a, b) provided that $| x_2 - x_1 | < \epsilon$.

Now $| \theta h | < | h |$; therefore

$$| f'(x+\theta h) - f'(x) | < \delta$$

if $| h | < \epsilon$.

(*g*) *Theorem G*. $f'(x)$ takes every value between its upper and lower boundaries in (a, b); in addition to the assumptions in Dini's form of the mean-value theorem, it being assumed that $f'(a+0)$ and $f'(b-0)$ exist.[1]

Proof. By *Theorem E*, $f'(x) \equiv \phi(x)$ is either a continuous function of x or, if discontinuous, has discontinuities of the *second* kind only. Therefore, in any case, $\phi(x)$ takes every value between its upper and lower boundaries; as required by Weierstrass's theorem about continuous functions and by Darboux's theorem about functions which have discontinuities of the second kind.

An *alternative proof*[2] is the following:—

Let U and L be the upper and lower boundaries of $f'(x)$ in (a, b). Then, if U > C > L, values x_1 and x_2 of x exist such that in a neighbourhood of x_1, say $(x_1 - h_1, x_1 + h'_1)$, the lower boundary of $f'(x)$ is L, and in a neighbourhood $(x_2 - h_2, x_2 + h'_2)$ of x_2 the upper boundary is U; so that $f'(x_1)$ is $L + \epsilon_1$ and $f'(x_2)$ is $U - \epsilon_2$, ϵ_1 and ϵ_2 being quantities greater than 0 but *as small as we please*.

[1] Given by Darboux in *Ann. de l'E'cole Normale Sup.*, Series 2, Vol. 4, 1875, p. 109.

[2] See Dini's *Calcolo Infinitesimale*, t. 1, pp. 54-55.

Now consider the function

$$\phi(x) \equiv f(x) - Cx.$$

Indicating with h a number > 0 but sufficiently small, we have obviously

$$\lim \frac{\phi(x_1+h)-\phi(x_1)}{+h} < 0$$

$$\lim \frac{\phi(x_2-h)-\phi(x_2)}{-h} > 0.$$

Hence there must exist a number $\delta > 0$ such that, for values of x between x_1 and $x_1+\epsilon$ (x_1 excluded) and for values of x between $x_2-\epsilon$ and x_2 (x_2 excluded), we shall have respectively the inequalities

$$\phi(x)-\phi(x_1) < 0,$$

$$\phi(x)-\phi(x_2) < 0.$$

The above inequalities show that neither x_1 nor x_2 is a minimum of $\phi(x)$. Now, $\phi(x)$ being a continuous function in (x_1, x_2), there must be a value x_0 inside (x_1, x_2) where the minimum of $\phi(x)$ is attained. Therefore, for all positive values of h less than a given number,

$$\phi(x_0+h)-\phi(x_0) \geq 0,$$

$$\phi(x_0-h)-\phi(x_0) \geq 0\ ;$$

whence

$$\frac{\phi(x_0+h)-\phi(x_0)}{h} \geq 0,$$

$$\frac{\phi(x_0-h)-\phi(x_0)}{-h} \leq 0.$$

Therefore, as $\phi(x)$ has a differential coefficient at x_0, the two expressions on the right side of the above must tend to 0 with h. Thus $\phi'(x_0)=0$, *i.e.*, $f'(x_0)=C$, which proves the theorem.

(*h*) *Theorem H*. If $f(x)$ satisfies the conditions of *Theorem G*, then $f'(x)$ cannot pass from a value A to a value B without taking all the intermediate values.

The proof is similar to the proof given above of *Theorem G*.

THIRD LECTURE

Generalizations of the Mean-Value Theorem

§ 18.

28. To-day's lecture will deal chiefly with such results as establish the validity of the mean-value theorem

$$f(x+h)=f(x)+hf'(x+\theta h),\ 0<\theta<1,$$

under conditions less restrictive than those in Dini's form of the theorem; the mean-value theorems with such less restrictive conditions may be considered generalizations of the theorem as found in the most up-to-date text-books on the Differential Calculus. To-day I will also give a number of deductions from these generalizations. I proceed now to give the generalization of W. H. Young and G. C. Young which may be enunciated as follows:—

If in a given interval (a, b) a function $f(x)$ is defined to be finite and continuous, the end-points being included; then, for every pair of points (x_0, x_0+h) of (a, b), the ends being included,

$$f(x_0+h)=f(x_0)+hf'(x_0+\theta h),\ 0\leqq\theta\leqq 1,$$

provided that at every point inside (a, b), the ends being excluded, there is no distinction of right and left with respect to the four derivates at that point so that $D^+f(x)=D^-f(x)$ and $D_+f(x)=D_-f(x)$.

Proof:

As in Dini's proof of the mean-value theorem in Art. 23, let $\phi(x)$ denote

$$f(x)-f(x_0)-\frac{x-x_0}{h}\{f(x_0+h)-f(x_0)\}.$$

Then $\phi(x_0)$ and $\phi(x_0+h)$ both vanish. Also like $f(x)$, $\phi(x)$ is finite and continuous in (x_0, x_0+h) and does not show any distinction of right and left with regard to the derivates at any point inside (x_0, x_0+h); thus at every such point

$$D^+\phi(x)=D^-\phi(x)\ \text{and}\ D_+\phi(x)=D_-\phi(x).$$

Now, if $\phi(x)$ is not zero throughout (x_0, x_0+h) it must have an upper bound or a lower bound different from zero. Also, because of the continuity of $\phi(x)$, this upper or lower bound must be attained at an interval point of (x_0, x_0+h). Calling ξ such a point there must exist a number $\epsilon > 0$ but sufficiently small so that, taking h to be always >0,

$$\left.\begin{array}{l} \phi(\xi+h)-\phi(\xi)<0 \\ \phi(\xi-h)-\phi(\xi)<0 \end{array}\right\} \text{ for } h<\epsilon$$

or

$$\left.\begin{array}{l} \phi(\xi+h)-\phi(\xi)>0 \\ \phi(\xi-h)-\phi(\xi)>0 \end{array}\right\} \text{ for } h<\epsilon.$$

In the first alternative $D^+\phi(\xi)$ and $D_+\phi(\xi)$ are both ≤ 0; and $D^-\phi(\xi)$ and $D_-\phi(\xi)$ are both ≥ 0. In the second alternative the inequalities are all reversed. Therefore, because of the hypothesis of their being no distinction of right and left,

$$D^+\phi(\xi)=D^-\phi(\xi)=0,$$
$$D_+\phi(\xi)=D_-\phi(\xi)=0;$$

and, consequently, $\phi'(\xi)$ exists and equals zero. Therefore

$$f'(\xi)=\frac{f(x_0+h)-f(x_0)}{h},$$

$$\text{i.e.,}\quad f(x_0+h)=f(x_0)+h\,f'(x_0+\theta h),\ 0<\theta<1.$$

§ 10.

29. A few examples may be given to illustrate the above generalization.

Ex. 1. Let $f(x)$ be $\frac{x}{\sqrt{2}}\cos\left(\frac{1}{2}\log\frac{1}{x^2}+\frac{\pi}{4}\right)$ for $-1\leq x\leq 1$, then $f(x)$ is finite and continuous in the interval $(-1, 1)$ and has a differential co-efficient $\cos\left\{\frac{1}{2}\log\frac{1}{x^2}\right\}$ at every point with the exception of $x=0$ where, however, there is no distinction of right and left with respect to the derivates.[1] Although Dini's condition is not satisfied, the mean-value theorem holds, and for every pair of points (x_0, x_0+h) in $(-1, 1)$

$$f(x_0+h)-f(x_0)=hf'(x_0+\theta h),\ 0<\theta<1,$$

$$\text{i.e.,}\quad (x_0+h)\cos\left\{\frac{1}{2}\log\frac{1}{(x_0+h)^2}+\frac{\pi}{4}\right\}-x_0\cos\left(\frac{1}{2}\log\frac{1}{x_0{}^2}+\frac{\pi}{4}\right)$$

$$=h\cos\tfrac{1}{2}\log\frac{1}{(x_0+\theta h)^2},\ 0<\theta<1.$$

[1] Obviously, the same holds at every point where there is a differential co-efficient.

Ex. 2. Let $f(x)$ be $x \sin \frac{1}{x}$ for $-1 \leq x \leq 1$, then $f(x)$ is finite and continuous in the interval $(-1, 1)$ and has a differential co-efficient $\sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$ at every point with the exception of $x=0$ where, however, there is no distinction of right and left with regard to the derivates. Therefore, although Dini's condition is not satisfied, the condition of the Youngs is satisfied and the mean-value theorem holds.

§ 20.

30. The next generalization to be considered is the following due to A. N. Singh:—

If in a given interval (a, b) a function $f(x)$ is defined to be finite and continuous, the end-points being included; then, for every pair of points (x_0, x_0+h) of (a, b) the ends being included,

$$f(x_0+h)=f(x_0)+h f'(x_0+\theta h),\ 0 \lesssim \theta \lesssim 1,$$

provided that,

(*i*) there is no point inside (a, b) at which one of the two derivatives, *viz*., the progressive differential co-efficient and the regressive differential co-efficient, exists, while the other does not exist, and

(*ii*) at every point inside (a, b) the upper and lower derivates on one side lie within or are equal to the upper and lower derivates on the other side.

Proof:

Consider the function

$$\phi(x) \equiv f(x)-f(x_0)-\frac{x-x_0}{h}\left\{f(x_0+h)-f(x_0)\right\}.$$

Then obviously $\phi(x_0)$ and $\phi(x_0+h)$ both vanish; also $\phi(x)$ is like $f(x)$ finite and continuous in (x_0, x_0+h). Then, unless $\phi(x)$ is zero throughout the interval (x_0, x_0+h), $\phi(x)$ has at least one maximum or one minimum inside the interval.

Suppose $\phi(x)$ has a maximum at the point ξ within (x_0, x_0+h). Then, by the hypothesis (*ii*) of the theorem, the derivates at ξ satisfy either the inequalities

$$\left.\begin{aligned} D^+\phi(\xi) &\geq D^-\phi(\xi), \\ D_+\phi(\xi) &\leq D_-\phi(\xi), \end{aligned}\right\} \text{(A)}$$

or

$$\left.\begin{aligned} D^+\phi(\xi) &\leq D^-\phi(\xi), \\ D_+\phi(\xi) &\geq D_-\phi(\xi). \end{aligned}\right\} \text{(B)}$$

Suppose that the inequalities (A) hold. Now, as ξ is a point of maximum,

$$D^+\phi(\xi) \leq 0 \text{ and } D^-\phi(\xi) \geq 0. \qquad (a)$$

But by the inequalities (A).

$$D^+\phi(\xi) \geq D^-\phi(\xi).$$

Therefore, combining the above with (*a*), we have

$$D^+\phi(\xi) = D^-\phi(\xi) = 0. \qquad (b)$$

Again, $D_-\phi(\xi) \geq 0$, for ξ is a point of maximum.

But

$$D_-\phi(\xi) \leq D^-\phi(\xi).$$

i.e., because of (*b*)

$$D_-\phi(\xi) \leq 0.$$

Therefore

$$D_-\phi(\xi) = 0 = D^-\phi(\xi).$$

Hence

$\phi'(\xi-0)$ exists and is 0; also by (*b*) $D^+\phi(\xi)=0$.

Therefore $\phi'(\xi+0)$ exists by the hypothesis (*i*) of the theorem and must be 0. Thus $\phi'(\xi)$ exists and is zero.

Hence

$$f'(\xi) = \frac{f(x_0+h)-f(x_0)}{h},$$

$$i.e., \; f(x_0+h) = f(x_0) + h\,f'(x_0+\theta h), \; 0<\theta<1.$$

In a similar manner, if the derivates at ξ satisfy the inequalities (B), it can be shown that

$$D^+\phi(\xi) = D_+\phi(\xi) = D_-\phi(\xi) = 0,$$

so that by the hypothesis (*i*) $\phi'(\xi)$ exists and is equal to 0 and the theorem holds.

If ξ is a point of minimum, it can be shown similarly that the theorem holds.

Thus the theorem is completely established.

§ 21.

31. The following two examples illustrate Singh's generalization:—

Ex. 1. Let $f(x)$ equal $2x \sin\frac{1}{x}$ in the interval (0, 1) and equal $x \sin\frac{1}{x}$ in the interval $(-1, 0)$. Then at the point $x=0$ inside $(-1, 1)$ there is a distinction of right and left with regard to the derivates; for,

$$D^{+}f(0)=2,\ D_{+}f(0)=-2$$
$$D^{-}f(0)=1,\ D_{-}f(0)=-1,$$

so that

$$D^{+}f(0) \gtrless D^{-}f(0) \text{ and not equal; also}$$
$$D_{+}f(0) \text{ is unequal to } D_{-}f(0).$$

At all the other points the differential co-efficient exists and so there is no distinction of right and left. Also $f(x)$ is finite and continuous in $(-1, 1)$, the ends being included. Still, for every pair of points (x_0, x_0+h) *whatever*, the theorem holds.

Ex. 2. Let $\psi(x)$ equal $2x \cos\log(x^2)$ in the interval (0, 2) and be equal to $x \cos\log(x^2)$ in the interval $(-2, 0)$. Then, for

$$f(x)=\sum_{n=1}^{\infty}\frac{\psi(x-\omega_n)}{2^n},$$

where $\{\omega_n\}$ is an enumerable and everywhere dense set of points in $(-1, 1)$, the generalization of Singh holds; so that the mean-value theorem is valid, although at every point of the set $\{\omega_n\}$ there *is* a distinction of right and left with regard to the derivates.

§ 22.

32. Singh has generalized the mean-value theorem still further, by extending it to even certain types of *discontinuous* functions.[1] His second generalization runs as follows:

If inside a given interval (a, b) for which $f(x)$ is defined, there be a point X at which there is a discontinuity of the *second* kind, at least on one side, say the right (left), and if inside a finite interval, however small, with the point of discontinuity X as left (right) end-point,

[1] In his paper (l. c.) Hedrick shows that the mean-value theorem *may* hold for a discontinuous function like $\sin\frac{1}{x}$.

(*i*) there is no point at which one of the derivatives exists while the other does not, and

(*ii*) at every point inside that interval the upper and lower derivates on one side lie within or are equal to the upper and lower derivates on the other; then the mean-value theorem

$$f(x_0+h)=f(x_0)+h\,f'(x_0+\theta h),\ 0<\theta<1,$$

holds.

Proof:

If (x_0, x_0+h) does not include X as an inside or end-point then the proof of § 20 holds.

If (x_0, x_0+h) includes X as an inside or end-point, then we proceed as follows. As the conditions of the generalization of § 20 are satisfied in an interval (X, a) lying inside (x_0, x_0+h) it follows [1] that the differential co-efficient exists at an everywhere dense set in (X, a), and passes through all the values between the upper and lower limits of the incrementary ratio in (X, a). As there is a discontinuity of the second kind at X, the upper and lower limits of the incrementary ratio are $+\infty$ and $-\infty$ respectively. Therefore, there exists at least one point in (X, a), and consequently inside (x_0, x_0+h), at which the differential co-efficient exists and has any given value. Hence there is at least one value of θ for which

$$\frac{f(x_0+h)-f(x_0)}{h}=f'(x_0+\theta h),\ 0<\theta<1.$$

33. The following example of Singh illustrates his second generalization:

Let G be a non-dense set of points in (x_0, x_0+h). Let (α, β) be an interval complementary to the set G, and let $\phi(x, \alpha)$ denote $\sin\dfrac{1}{x-\alpha}$.

Further, let $\alpha+\gamma$ be the greatest value of x, which does not exceed $\frac{1}{2}(\alpha+\beta)$, for which $\cos\dfrac{1}{x-\alpha}$ vanishes. Let $F(x)$ be zero at every point of G. And in each interval (α, β) complementary to G, let

$$F(x)=\phi(x, \alpha),\ \text{for}\ \alpha<x<\alpha+\gamma;$$

$$F(x)=\phi(\alpha+\gamma, \alpha),\ \text{for}\ \alpha+\gamma\leq x\leq\beta-\gamma;$$

[1] By Theorem O of Article 37.

$$F(x) = -\phi(x, \beta), \text{ for } \beta - \gamma < x < \beta.$$

It is easy to see that any interval (p, q) taken within (x_0, x_0+h) contains within it a complementary interval of the set G, or is itself contained within such an interval. In the first case, the conditions of the theorem of Art. 32 are satisfied for the interval (p, q), and in the second case those of the theorem of Art. 30. Hence the mean-value theorem holds for any interval within (x_0, x_0+h) although $F(x)$ is discontinuous at all the points of the set G in (x_0, x_0+h).

§ 23.

34. Because of its historical interest as showing Dini's attempt at a generalization [1] of the mean-value theorem thirty years before the publication of the generalization of the Youngs, the following is reproduced almost word for word from Dini's "*Fondamenti.*"

" If a function $f(x)$ is finite and continuous in the whole of an interval, and, excepting only a finite number of points $a_1, a_2, a_3, \ldots$ at which there is uncertainty, in all the other points of the interval it has a determinate differential co-efficient which never exceeds a finite number, and for the points $a_1, a_2, a_3, \ldots$ the ratios

$$\frac{f(a_1+\delta)-f(a_1)}{\delta}, \quad \frac{f(a_2+\delta)-f(a_2)}{\delta}, \ldots$$

with the numerical diminution of δ do not tend to a finite and determinate limit but oscillate only between finite limits ; then, denoting by x, $x+h$ two points whatever of the given interval, we shall have

$$f(x+h)-f(x) = h\,A,$$

where A is a quantity which depends on x and on h but which in absolute value never exceeds a given finite and positive number.

Let us observe in fact that, if between x and $x+h$ (x and $x+h$ being at the most excluded) $f(x)$ has always a differential co-efficient then this theorem follows quickly " from the ordinary theorem given in Art. 23 ; " so that it is necessary now to consider the case in which one or more of the points $a_1, a_2, a_3, \ldots$ fall in the *interior* of the interval from x to $x+h$.

[1] See No. 7, pp. 75-76 of the book of Dini.

Now in this case, by the hypotheses made, there shall exist a number δ_1 positive and different from zero such that for δ numerically inferior to δ_1 and for $\delta = \pm\delta_1$ the ratios

$$\frac{f(a_1+\delta)-f(a_1)}{\delta},\quad \frac{f(a_2+\delta)-f(a_2)}{\delta},\ldots$$

are *all* numerically less than a finite quantity g, while for δ numerically greater than δ_1 the same ratios shall be all inferior in absolute value to $\frac{2g'}{\delta}$, g' being the maximum absolute value of $f(x)$ in the given interval ; whence, denoting by A' the greater of the quantities g and $\frac{2g'}{\delta_1}$, and supposing that in the interior of the interval from x to $x+h$ falls at least one of the points $a_1, a_2, a_3, \ldots$, *e.g.*, the point a_1, in such a manner that we can put

$$x = a_1 \mp \delta,\ x+h = a_1 \pm \delta',$$

where δ and δ' are positive, we shall have in absolute value

$$f(a_1 \pm \delta') - f(a_1) < \delta' A',\ f(a_1 \mp \delta) - f(a_1) < \delta A',$$

and hence also in absolute value

$$f(a_1 \pm \delta') - f(a_1 \mp \delta) < (\delta + \delta')A',$$

and consequently

$$f(x+h) - f(x) = hh_1 A',$$

h_1 being a quantity numerically inferior to 1; and this evidently proves the theorem."

35. Something which may be regarded as a generalization in *appearance* only was given by Harnack and Thomae [1] independently of each other. It may be stated as follows: If in an interval (a, b), $f(x)$ is defined as a finite and continuous function and if one of the four derivates, say the upper derivate on the right, is a continuous function of x inside the interval ; then for any two points x_0, x_0+h in (a, b), $f(x_0+h) = f(x_0) + h\,f'(x_0+\theta h)$, $0<\theta<1$.

The continuity of the particular derivate carries with it the existence of the differential co-efficient and its equality with the derivate. So really the theorem is no generalization.

[1] Harnack : *Die Elemente der Differential- u. Integralrechnung*, p 181 ; Thomae : *Einleitung in die Theorie der bestimmten Integrale*, p. 10.

§ 24.

36. The following two theorems are worthy of mention as generalizations of Rolle's theorem:

(a) *Rolle's theorem for derivates* [1]: If $f(x)$ is a function of x which is finite and continuous in an interval (a, b), the ends being included, and is zero at the end-points a and b; then there is a point ξ inside the interval (a, b), the ends being excluded, at which one of the upper derivates is not positive and the other lower derivate is not negative, that is

$$D^+f(\xi) \leq 0 \leq D_-f(\xi),$$

or the alternative inequality, interchanging left and right,

$$D^-f(\xi) \leq 0 \leq D_+f(\xi).$$

Proof:

If $f(x)=0$ throughout the whole interval, the theorem is evident, all the derivates being zero everywhere. If not, $f(x)$ has a positive upper bound or a negative lower bound, or both, and, being a continuous function, there is a point ξ inside (a, b) at which $f(x)$ assumes such an extreme value.

It then follows as in Art. 28 that

$$\frac{f(\xi+h)-f(\xi)}{h}, \quad \frac{f(\xi-h)-f(\xi)}{-h}$$

are respectively ≤ 0 and ≥ 0 if $f(\xi)$ is a maximum; the signs in the inequalities are respectively ≥ 0 and ≤ 0 if $f(\xi)$ is a minimum.

Therefore, proceeding to the limit, it follows that the derivates on one side of ξ are ≤ 0 and on the other ≥ 0 which proves the theorem.

(b) *Rolle's theorem for derivatives* [2]: If $f(x)$ be a finite and continuous function defined in an interval (a, b), the ends being included, and be such that the upper and lower derivates on one side lie within or are equal to the upper and lower derivates on the other side; then there is a point ξ inside the interval, the ends being excluded, where at least one of the two

$$f'(\xi+0), \; f'(\xi-0)$$

exists and equals $\quad \dfrac{f(b)-f(a)}{b-a}$.

[1] Given by the Youngs (see *l. c.*, foot-note of p. 8).

[2] Given by Singh (*l. c.*, Theorem II of p. 46). The proof is contained in the proof given of the generalization in Art. 30.

37. I will conclude this lecture by giving a number of deductions from considerations similar to those in the generalizations given to-day.

Theorem I.[1]

If y be a continuous function of x in the interval (x_0, x_1), and if the upper derivate to the right lies everywhere between the limits G and G′ (G>G′), then the quotient

$$\frac{y'-y}{x'-x}$$

also lies between the same limits, whatever arbitrary values be taken for x and x' in (x_0, x_1).

Proof:

Let $x'>x$. If $\frac{y'-y}{x'-x}$ were $>$ G, then a quantity $c>0$ could be chosen to be so small that, if z were defined by

$$z=y-(\mathrm{G}+c)x,$$

the difference $z'-z$ would be >0, say equal to δ. Therefore, then must exist for x an upper bound x'' $(<x')$ where the relation

$$z'-z''=\delta$$

would be satisfied last. At this place would

$$\mathrm{D}^{+}z''\geq 0,$$

and, consequently,

$$\mathrm{D}^{+}y''\geq \mathrm{G}+c,$$

which would be against the hypothesis.

If, however, $\frac{y'-y}{x'-x}$ were $<$ G′, then a quantity $c>0$ could be chosen to be so small that, if z were defined by

$$z=y-(\mathrm{G}'-c)x,$$

the difference $z'-z$ would be <0, say equal to $-\delta$. Therefore, then must exist for x an upper bound x'' $(<x')$, where the relation

$$z'-z''=-\delta$$

would be satisfied last. At this place would

$$\mathrm{D}^{+}z''\leq 0$$

or

$$\mathrm{D}^{+}y''\leq \mathrm{G}'-c,$$

what is again against the hypothesis.

If in the theorem, instead of the upper derivate on the right, we take any of the other derivates, the theorem will remain true and the proof will be similar to that given above.

[1] The enunciation as well as the proof is almost word for word the same as in Scheefer's paper in *Acta Math.*, Vol. 5 (1884), p. 190.

Theorem J: For a continuous function, the four derivates all have the same upper bound and the same lower bound in a given interval, and these upper and lower bounds are also the upper and lower bounds of the difference quotient

$$\frac{f(x_2)-f(x_1)}{x_2-x_1},$$

where x_1, x_2 take all possible values in the given interval.

Proof:

Let U and L denote respectively the upper bound and the lower bound of

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}.$$

Then it is obvious from Theorem I that

$$U \leq G, \ L \leq G'.$$

We proceed to show that in the above the only signs permissible are those of equality.

For, if possible, suppose for example that $U<G$ and is equal, say, to $G-\eta$ where $\eta>0$. Thus it is not possible to find two values x_1, x_2, which will give

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}>G-\eta.$$

But this is absurd, because from the fact that the upper bound of $D^+f(x)$ is G, it follows that however small a positive number ϵ may be taken, there are values x_1 and h such that

$$\frac{f(x_1+h)-f(x_1)}{h}>G-\epsilon.$$

Similarly, it can be proved that L cannot be $<G'$. Thus it is proved that the upper derivate and the incrementary ratio have the same upper and lower bounds.

In the same manner, it can be proved that any of the other derivates has the same upper and lower bounds as the incrementary ratio.

Theorem K: If one of the derivates of a continuous function has a positive lower bound, then the function is monotone and increasing; if one of the derivates of a continuous function has a negative upper bound, then the function is monotone and decreasing.

This follows inumerately from Theorem J.

Theorem L: If a continuous function $f(x)$ has a derivate which is continuous at a point a, then all the derivates are continuous at a and $f(x)$ has a differential co-efficient at that point.

Proof:

Take any interval $(a-\epsilon_0, a+\epsilon_1)$ round the point a. Then in this interval the upper and lower bounds of the derivate, say $D^+f(x)$, are also respectively the upper and lower bounds of any of the three remaining derivates. As $D^+f(a)$ is finite, the upper and lower bounds of $D^+f(x)$ in $(a-\epsilon_0, a+\epsilon_1)$ each differ from $D^+f(a)$ by a number which with the diminution of ϵ_0 and ϵ_1 tends to 0. The above remark holds also for the other three derivates. Therefore all the four derivates are continuous at a; hence they are all equal at a and there is a differential co-efficient $f'(a)$.

Theorem M: A function is determined except as to an additive constant if we know a finite derivate of the function for all values of the variable.

Proof [1]:

Let $f(x)$ and $F(x)$ be two functions which have the same derivate, say the upper derivate on the right. Then form the function

$$\phi(x) \equiv cx + F(x) - f(x),$$

where $c>0$ is an arbitrarily chosen quantity.

(a) It can be proved as follows that

$$D^+\phi(x) \geq c.$$

If δ be a second positive quantity, then, for every value of x, arbitrarily small but positive values of h exist for which

$$\frac{F(x+h)-F(x)}{h} > D^+F(x) - \delta.$$

Also for *all* positive values of h lower than a certain quantity

$$\frac{f(x+h)-f(x)}{h} < D^+f(x) + \delta.$$

Therefore it follows that arbitrarily small values of h exist for which

$$\frac{F(x+h)-F(x)}{h} - \frac{f(x+h)-f(x)}{h} > -2\delta$$

i.e.,
$$\frac{\phi(x+h)-\phi(x)}{h} > c - 2\delta.$$

[1] The proof is taken almost word for word from Scheefer's paper in *Acta Math.*, Vol. 5, pp. 184-185.

Hence $D^+\phi(x) \geq c - 2\delta$ and consequently, since δ was chosen to be arbitrary,

$$D^+\phi(x) \geq c.$$

(β) If the domain of x be (a, b), it can be proved as follows that in (a, b) $\phi(x) - \phi(a)$ can never be <0. For if it were <0 at a point x', then would the values of x between a and x', for which $\phi(x) - \phi(a)$ would be ≥ 0, have an upper bound, say, x''. As $\phi(x) - \phi(a)$ is continuous must

$$\phi(x'') - \phi(a) = 0,$$

and for every value of $h < x' - x''$ the relation

$$\phi(x'' + h) - \phi(x'') < 0$$

must hold which would be untenable with the result proved under (α) that $D^+\phi(x) \geq c$.

Hence $\{F(x) - f(x)\} - \{F(a) - f(a)\}$ can for no value of x be <0. For, otherwise, c could be chosen to be so small that also $\phi(x) - \phi(a)$ would be <0 which is impossible.

(γ) Interchanging $f(x)$ and $F(x)$ and by reasoning as above, it follows that

$$\{f(x) - F(x)\} - \{f(a) - F(a)\} \text{ can for no value of } x \text{ be} < 0,$$

i.e., $\{F(x) - f(x)\} - \{F(a) - f(a)\}$ can for no value of x be >0.

But it has been proved under (β) that the above expression can for no value of x be <0.

Hence it must be equal to 0.

Theorem N: If there is no distinction of right and left with regard to the derivates of $f(x)$ in the interval (a, b) in which $f(x)$ is defined as a continuous function, then

(*i*) the points where $f(x)$ has a differential co-efficient $f'(x)$ are dense everywhere, and of potency c;

(*ii*) the differential co-efficient assumes every value between its upper and lower bounds in any closed interval at points internal to the interval;

(*iii*) the upper and lower bounds of $f'(x)$ in any interval, open or closed, are the same as those of the incrementary ratio.

Proof:

(α) By Theorem J, the upper and lower bounds of any derivate are respectively U and L, the upper and lower bounds of the incrementary ratio. Now let K be a number such that

$$L < K < U.$$

Then, as proved below under (γ), there is at least one interval (a_k, b_k) within (a, b) such that

$$\frac{f(b_k)-f(a_k)}{b_k-a_k}=\mathrm{K}.$$

And, as the conditions for the generalization of the Youngs hold, there exists a point x_k in (a_k, b_k) such that there is a differential co-efficient $f'(x_k)$ of $f(x)$ at x_k equal to

$$\frac{f(b_k)-f(a_k)}{b_k-a_k}, \; i.e., \mathrm{K}.$$

But K may have any value between L and U, so that the points x_k where the differential co-efficient of $f(x)$ exists, are in (1, 1) correspondence with the continuum (L, U). Again, obviously x_k exists in any interval ever so small. Thus it is proved that $\{x_k\}$ is everywhere dense and of potency c.

(β) The reasoning in (α) also shows that $f'(x)$ assumes every value between its upper and lower bounds in any closed interval at points internal to the interval.

(γ)[1] Consider the incrementary ratio

$$m(x, x')=\frac{f(x)-f(x')}{x-x'}$$

where $f(x)$ is a finite and continuous function at points for which

$$a<x<x'<b.$$

Consider the plane function $m(x, y)$. This is definite and continuous at every point of the closed square bounded by the lines

$$x=a, \; x=b,$$
$$y=a, \; y=b,$$

excepting only on the diagonal $x=y$.

Moreover, since identically

$$m(x, y)=m(y, x)$$

the values assumed by $m(x, y)$, in one of the triangles into which the square is divided by its diagonal are the same as in the other, so that all possible values of $m(x, y)$ in the square are assumed inside and on the two equal sides of the triangle throughout whose interior $x<y$.

These values are clearly none other than the values of $m(x, x')$ where x and x' are two distinct points of the closed interval (a,b).

Let k be any value between the upper and lower bounds (non-inclusive) of $m(x, x')$ in the closed interval (a, b). Then we can find two values k_1 and k_2 both of which are assumed by the incrementary ratio, where

[1] The proof is taken from the paper of the Youngs, *l.c.*, pp. 5-7.

$$k_1 < k < k_2.$$

By what precedes there is a point P_1 inside the triangle or on one of its equal sides at which

$$m(x, y) = k_1.$$

Similarly there is a point at which

$$m(x, y) = k_2,$$

if this and P_1 are not both on the same side of the triangle let this be P_2. If, however, P_1 and this point lie on one and the same side of the triangle we may take as P_2 a point near the point found, $m(x, y)$ being continuous there, the value k'_2 of $m(x, y)$ at P_2 being such that still

$$k_1 < k < k'_2.$$

The stretch P_1 P_2 will then lie inside the triangle with at most its end-points P_1 and P_2 on the boundary, but not on the same side of the triangle, and the stretch will not meet the hypotenuse. Thus $m(x, y)$ is finite at every internal point of the stretch P_1 P_2, and has at the end-points the finite or infinite values k_1 and k_2. Since $m(x, y)$ is continuous at every point of the closed stretch P_1P_2 it follows that it assumes the value k lying between its values at P_1 and P_2 at some intermediate point, that is at some point internal to the triangle.

Theorem O: If $f(x)$ satisfies the conditions for Singh's first generalization, given in Art. 30, *viz.*, $f(x)$ is continuous in a given interval (a, b), there is no point inside (a, b), at which one of the two derivatives exists and the other does not exist, and at every point inside (a, b) the upper and lower derivates on one side lie within or are equal to the upper and lower derivates on the other side; then there exists an everywhere dense set of points of potency c, where the differential co-efficient of $f(x)$ exists, and this differential co-efficient passes through every value between its upper and lower limits.

Proof:

Reasoning as under (α) and (β) above, the theorem is proved.

Theorem P: If the second of the three conditions of the above theorem is not satisfied and the other two are satisfied; then there is an everywhere dense set of points of potency c at which one of the derivatives exists, and the derivative passes through all the values between its upper and lower limits.

Proof:

The reasoning under (α) and (β) may be easily modified by using the word *derivative* instead of *differential co-efficient*, because of the validity of the result (A), (B), (a) and (b) of page 36. The theorem is therefore proved.

FOURTH LECTURE

THE FUNCTIONAL NATURE OF θ

§ 25

38. Before I proceed with an account of the recent researches on the functional nature of θ, I will consider, because of its historical interest, the following problem studied by Whitcom fifty years ago in what may be rightly believed to be the first publication [1] on the functional nature of θ:—

Assuming (*i*) that $f'(x+\theta h)$ is expansible in powers of h in the series

$$f'(x+\theta h) = \sum_0^\infty \frac{f^{(n+1)}(x)}{n!}\theta^n h^n$$

and $f(x+h)$ in the series

$$f(x+h) = \sum_0^\infty \frac{f^{(n)}(x)}{n!}h^n,$$

for a certain domain of h including o and (*ii*) that θ is also expansible in powers of h in the series

$$\theta = \sum_0^\infty A_n h^n,$$

determine the A's, it being further assumed that $f^{(2)}(x) \neq o$.

Whitcom has attempted the solution of the problem and given the actual values of $A_0, A_1, A_2, A_3, A_4, A_5$; B. N. Pal [2] has determined A_6 and A_7. The following method is substantially the same as Edwards' [3] and Pal's and similar to Whitcom's:

Since

$$f(x+h) = f(x) + h\, f'(x+\theta h),$$

$$\sum_o^\infty \frac{f^{(n)}(x)}{n!}h^n \equiv f(x) + h\sum_{m=0}^\infty \frac{f^{(m+1)}(x)}{m!}h^m\left\{\sum_{r=0}^\infty A_r\, h^r\right\}^m \qquad \dots \quad (1)$$

[1] "On the expansion of $\phi(x+h)$" (*American Journal of Math.*, Vol. 3, 1880, pp. 344-355).

[2] "On the expansion of θ in the mean-value theorem of the Differential Calculus" (*Bulletin of the Calcutta Mathematical Society*, Vol. XIX, 1928, pp. 143-146).

[3] "Treatise on the Differential Calculus," pp. 103-104; also p. 110. Only A_0, A_1, A_2, A_3 are correctly given but no mention is made of Whitcom.

Now equate the co-efficients of h^{n+2} on both the sides of (1), then

$$\frac{f^{(n+2)}(x)}{(n+2)!}=\sum_{m=1}^{n+1}\frac{f^{(m+1)}(x)}{m!}\,\mathrm{T}_{n+1-m,\ m},\quad n\geq 0, \qquad \ldots\ (2)$$

where $\mathrm{T}_{N,K}$ [1] denotes the co-efficient of h^N in the expansion of

$$\left\{\sum_{r=0}^{\infty} \mathrm{A}_r\, h^r\right\}^{\mathrm{K}}.$$

From (2) we get the following equations first given by Whitcom[2]:—

$$\frac{1}{2!}f^{(2)}(x)=\mathrm{A}_0\, f^{(2)}(x),$$

$$\frac{1}{3!}f^{(3)}(x)=\mathrm{A}_1\, f^{(2)}(x)+\frac{\mathrm{A}_0^{\,2}}{2!}f^{(3)}(x),$$

$$\frac{1}{4!}f^{(4)}(x)=\mathrm{A}_2\, f^{(2)}(x)+\frac{2\mathrm{A}_0\mathrm{A}_1}{2!}\,f^{(3)}(x)+\frac{\mathrm{A}_0^{\,3}}{3!}\,f^{(4)}(x),$$

$$\frac{1}{5!}f^{(5)}(x)=\mathrm{A}_3\, f^{(2)}(x)+\frac{2\mathrm{A}_0\mathrm{A}_2}{2!}f^{(3)}(x)+\frac{\mathrm{A}_1^{\,2}}{2!}f^{(3)}(x)+\frac{3\mathrm{A}_0^{\,2}\mathrm{A}_1}{3!}f^{(4)}(x)$$

$$+\frac{\mathrm{A}_0^{\,4}}{4!}f^{(5)}(x),$$

$$\frac{1}{6!}f^{(6)}(x)=\mathrm{A}_4\, f^{(2)}(x)+\frac{2\mathrm{A}_0\mathrm{A}_3}{2!}f^{(3)}(x)+\frac{2\mathrm{A}_1\mathrm{A}_2}{2!}f^{(3)}(x)+\frac{3\mathrm{A}_0^{\,2}\mathrm{A}_2}{3!}\,f^{(4)}(x)$$

$$+\frac{3\mathrm{A}_0\mathrm{A}_1^{\,2}}{3!}\,f^{(4)}(x)+\frac{4\mathrm{A}_0^{\,3}\mathrm{A}_1}{4!}\,f^{(5)}(x)+\frac{\mathrm{A}_0^{\,5}}{5!}\,f^{(6)}(x),$$

$$\frac{1}{7!}f^{(7)}(x)=\mathrm{A}_5\, f^{(2)}(x)+\frac{2\mathrm{A}_0\mathrm{A}_4}{2!}f^{(3)}(x)+\frac{2\mathrm{A}_1\mathrm{A}_3}{2!}f^{(3)}(x)+\frac{\mathrm{A}_2^{\,2}}{2!}f^{(3)}(x)$$

$$+\frac{3\mathrm{A}_0^{\,2}\mathrm{A}_3}{3!}\,f^{(4)}(x)+\frac{\mathrm{A}_1^{\,3}}{3!}f^{(4)}(x)+\frac{6\mathrm{A}_0\mathrm{A}_1\mathrm{A}_2}{3!}\,f^{(4)}(x)$$

[1] For large values of N the expression for $\mathrm{T}_{N,K}$ is rather complicated. See recurrence formulae for its calculation in Glaisher's paper in *Q. J. M.*, Vol. 14, 1875, pp. 79-84; also see B. Hansted's paper in *Tidskrift for Mathematik*, Vol. 5, 1881, pp. 12-16.

[2] *l.c.*, p. 348.

$$+\frac{4A_0{}^3A_2}{4!}f^{(5)}(x)+\frac{6A_0{}^2A_1{}^2}{4!}f^{(5)}(x)+\frac{5A_0{}^4A_1}{5!}f^{(6)}(x)$$

$$+\frac{A_0{}^6}{6!}f^{(7)}(x).$$

$$\frac{1}{8!}f^{(8)}(x)=A_6f^{(2)}(x)+\frac{2A_0A_5}{2!}f^{(3)}(x)+\frac{2A_1A_4}{2!}f^{(3)}(x)+\frac{2A_2A_3}{2!}f^{(3)}(x)$$

$$+\frac{3A_0{}^2A_4}{3!}f^{(4)}(x)+\frac{3A_1{}^2A_3}{3!}f^{(4)}(x)+\frac{6A_0A_1A_3}{3!}f^{(4)}(x)$$

$$+\frac{3A_0A_2{}^2}{3!}f^{(4)}(x)+\frac{4A_0A_1{}^3}{4!}f^{(5)}(x)+\frac{4A_0{}^3A_3}{4!}f^{(5)}(x)$$

$$+\frac{12A_0{}^2A_1A_2}{4!}f^{(5)}(x)+\frac{5A_0{}^4A_2}{5!}f^{(6)}(x)+\frac{10A_0{}^3A_1{}^2}{5!}f^{(6)}(x)$$

$$+\frac{6A_0{}^5A_1}{6!}f^{(7)}(x)+\frac{A_0{}^7}{7!}f^{(8)}(x).$$

Two more equations have been given by Pal.[1] *viz.*, those corresponding to

$$\frac{1}{9!}f^{(9)}(x) \quad \text{and} \quad \frac{1}{10!}f^{(10)}(x).$$

From the first equation we get A_0, using the value of A_0 in the second we get A_1 and so on.

The actual values of the first eight A's are

$$A_0=\frac{1}{2},\quad A_1=\frac{1}{24}\frac{f^{(3)}}{f^{(2)}},\quad A_2=\frac{1}{2}\left[\frac{1}{24}\frac{f^{(4)}}{f^{(2)}}-\frac{1}{24}\frac{\{f^{(3)}\}^2}{\{f^{(2)}\}^2}\right],$$

$$A_3=\frac{1}{3!}\left[\frac{11}{320}\frac{f^{(5)}}{f^{(2)}}-\frac{3}{32}\frac{f^{(3)}f^{(4)}}{\{f^{(2)}\}^2}+\frac{11}{192}\frac{\{f^{(3)}\}^3}{\{f^{(2)}\}^3}\right],$$

$$A_4=\frac{1}{4!}\left[\frac{13}{480}\frac{f^{(6)}}{f^{(2)}}-\frac{43}{480}\frac{f^{(3)}f^{(5)}}{\{f^{(2)}\}^2}+\frac{7}{32}\frac{\{f^{(3)}\}^2f^{(4)}}{\{f^{(2)}\}^3}-\frac{1}{16}\left\{\frac{f^{(4)}}{f^{(2)}}\right\}^2\right.$$

$$\left.-\frac{3}{32}\left\{\frac{f^{(3)}}{f^{(2)}}\right\}^4\right],$$

[1] *l.c.*, pp. 144-45.

$$A_5=\frac{1}{5!}\left[\frac{19}{896}\frac{f^{(7)}}{f^{(2)}}-\frac{31}{384}\frac{f^{(3)}f^{(6)}}{\{f^{(2)}\}^2}+\frac{15}{64}\frac{\{f^{(3)}\}^2f^{(5)}}{\{f^{(2)}\}^3}-\frac{55}{108}\frac{\{f^{(3)}\}^3f^{(4)}}{\{f^{(2)}\}^4}\right.$$

$$\left.+\frac{5}{16}\frac{f^{(3)}\{f^{(4)}\}^2}{\{f^{(2)}\}^3}-\frac{53}{384}\frac{f^{(4)}f^{(5)}}{\{f^{(2)}\}^2}+\frac{185}{1152}\frac{\{f^{(3)}\}^5}{\{f^{(2)}\}^5}\right],$$

$$A_6=\frac{1}{6!}\left[\frac{15}{896}\frac{f^{(8)}}{f^{(2)}}-\frac{1}{14}\frac{f^{(3)}f^{(7)}}{\{f^{(2)}\}^2}+\frac{15}{64}\frac{\{f^{(3)}\}^2f^{(6)}}{\{f^{(2)}\}^3}-\frac{331}{576}\frac{\{f^{(3)}\}^2f^{5}}{\{f^{(2)}\}^4}\right.$$

$$+\frac{1175}{1152}\frac{\{f^{(3)}\}^4f^{(4)}}{\{f^{(2)}\}^5}-\frac{205}{192}\frac{\{f^{(3)}\}^2\{f^{(4)}\}^2}{\{f^{(2)}\}^4}+\frac{5}{32}\frac{\{f^{(4)}\}^3}{\{f^{(2)}\}^3}$$

$$\left.+\frac{47}{64}\frac{f^{(3)}f^{4}f^{5}}{\{f^{(2)}\}^3}-\frac{11}{128}\frac{\{f^{(5)}\}^2}{\{f^{(2)}\}^2}-\frac{9}{64}\frac{f^{(4)}f^{(6)}}{\{f^{(2)}\}^2}-\frac{85}{384}\left\{\frac{f^{(3)}}{f^{(2)}}\right\}^6\right],$$

$$A_7=\frac{1}{7!}\left[\frac{247}{18432}\frac{f^{(9)}}{f^{(2)}}-\frac{97}{1536}\frac{f^{(3)}f^{(8)}}{\{f^{(2)}\}^2}+\frac{703}{3072}\frac{\{f^{(3)}\}^2f^{(7)}}{\{f^{(2)}\}^3}-\frac{5551}{9216}\right.$$

$$\times\frac{\{f^{(3)}\}^3f^{(6)}}{\{f^{(2)}\}^4}+\frac{55363}{55296}\frac{\{f^{(3)}\}^4\{f^{(5)}\}}{\{f^{(2)}\}^5}-\frac{15995}{27648}\frac{\{f^{(3)}\}^5\{f^{(4)}\}}{\{f^{(2)}\}^6}$$

$$+\frac{4375}{2304}\frac{\{f^{(3)}\}^3\{f^{(4)}\}^2}{\{f^{(2)}\}^5}-\frac{245}{768}\frac{f^{(3)}\{f^{(4)}\}^3}{\{f^{(2)}\}^4}+\frac{287}{512}\frac{\{f^{(4)}\}^2\{f^{(5)}\}}{\{f^{(2)}\}^3}$$

$$-\frac{8253}{3072}\frac{\{f^{(3)}\}^2f^{(4)}f^{5}}{\{f^{(2)}\}^4}+\frac{14119}{30720}\frac{f^{(3)}\{f^{(5)}\}^2}{\{f^{(2)}\}^3}+\frac{35}{64}\frac{f^{(3)}f^{(4)}f^{(6)}}{\{f^{(2)}\}^3}$$

$$\left.-\frac{595}{3072}\frac{f^{(5)}f^{(6)}}{\{f^{(2)}\}^2}-\frac{71}{512}\frac{f^{(4)}f^{(7)}}{\{f^{(2)}\}^2}-\frac{2695}{18432}\left\{\frac{f^{(3)}}{f^{(2)}}\right\}^7\right].$$

§ 26

39. The next problem, after Whitcom's, goes back, according to Professor R. Rothe,[1] to the time of Cauchy, and may be enunciated as follows: Prove that $\theta(+o)$, *i.e.*, $\lim_{h=+o}\theta$, is $\frac{1}{2}$.

[1] See his paper, "Zum Mittelwertsatze der Differentialrechnung" (*Math. Zeit.*, Bd. 9, 1921, pp. 300-25), specially p. 307.

The following solution[1] due to Dini was given by him in 1907 and has the merit not only of being the oldest *published*[2] solution, leaving aside the fact that it may be regarded as contained in Whitcom's solution of the first problem, but also of being based on the sole assumptions (*i*) of the existence of $f^{(2)}(t)$ in a small interval $x \leq t \leq x+r$ and (*ii*) of $f^{(2)}(x)$ being different from o and infinity, and of not postulating the continuity of $f^{(2)}(t)$, in addition to the *tacit* assumption[3] of the single-valuedness of θ as a function of h.

In Cauchy's generalized mean-value theorem

$$\frac{\phi(x+h)-\phi(x)}{F(x+h)-F(x)}=\frac{\phi'(x+\theta_1 h)}{F'(x+\theta_1 h)}\ , \ o<\theta_1<1. \quad (M').$$

put

$$F(t)=(t-x),\ \phi(t)=f(t)-f(x)-(t-x)f'(x)-\frac{(t-x)^2}{2}f''(x).$$

Then (M') gives

put $t=x+h$

$$f(x+h)-f(x)-h\,f'(x)-\frac{h^2}{2}f''(x)=h\{f'(x+\theta_1 h)-f'(x)-\theta_1 h f''(x)\},$$

$$\text{i.e.,}\quad f(x+h)-f(x)-h\,f'(x)-\frac{h^2}{2}f''(x)$$

$$=h^2\theta_1\left\{\frac{f'(x+\theta_1 h)-f'(x)}{\theta_1 h}-f''(x)\right\}. \qquad (1)$$

[1] "Calcolo Infinitesimale," t. 1, 1907, p. 89 (foot-note).

[2] In Laurent's "Traité d' Analyse," t. 1, 1885, the following two questions appear on page 95 without any solution :—" 17. In the formula of Taylor $f(x+h)=f(x)+hf'(x)+\ldots+\frac{h^n}{n!}f^{(n)}(x+\theta h)$ the quantity θ is of the form $\frac{1}{n+1}+\epsilon$, ϵ denoting a quantity which becomes infinitely small with h, but which involves the existence of the $(n+1)$th differential co-efficient of f: by supposing the existence of the following differential co-efficients of f, one proposes to show that ϵ is of the form $hA+h^2B+\ldots$; A, B,... are functions which one proposes to calculate. (One demands only the values of the first quantities and not their general expansion.) 18. One has $f(x+h)=f(x)+hf'\left(x+\frac{h}{2}\right)$ to terms of the third order with respect to h."

[3] The mere existence and finiteness of $f^{(2)}(x)$ and its being $\neq o$ does *not* suffice for the single-valuedness of θ. Take, *e.g.*, $f(t)=\int_x^{t}\left\{\frac{1}{2}+\cos\frac{1}{z-x}\right\}dz$. Then $f^{(2)}(x)=\frac{1}{2}$, but $f(t)$ has infinite number of maxima and minima in the neighbourhood of x, *viz.*, those given by $\frac{1}{t-x}=2n\pi\pm\frac{2\pi}{3}$. So $\theta(h)$ has infinite number of values for a given h.

Now $f''(x)$ is finite according to hypothesis; therefore, with h tending to zero, the factor of h^2 in the left side of (1) tends to 0; put it equal to $\frac{\epsilon}{2}$, where ϵ tends to 0 with h.

Thus (1) may be written.

$$f(x+h)=f(x)+h f'(x)+\frac{h^2}{2}\left\{f''(x)+\epsilon\right\}. \tag{2}$$

But by the mean-value theorem

$$f(x+h)=f(x)+hf'(x+\theta h).$$

Therefore, using (2) with the above,

$$f'(x+\theta h)=f'(x)+\frac{h}{2}\left\{f''(x)+\epsilon\right\},$$

i.e.,
$$\frac{f'(x+\theta h)-f'(x)}{\theta h}=\frac{1}{2\theta}\left\{f''(x)+\epsilon\right\}. \tag{3}$$

Now as h tends to 0, the left side of (2) tends to $f''(x)$; put it equal to $f''(x)+\epsilon_1$, where ϵ_1 tends to 0 with h. Therefore (3) may be written in the form

$$f''(x)+\epsilon_1=\frac{1}{2\theta}\left\{f''(x)+\epsilon\right\}.$$

But, by hypothesis, $f''(x)\neq 0$; therefore, dividing both the sides of the above equation and making h tend to 0, we have

$$2\,\theta(+0)=1,\ i.e.,\ \theta(+0)=\tfrac{1}{2}.$$

40. *Rothe* has given a solution[1] of the problem, enunciated in Art. 39, by making the additional assumption of the continuity of $f''(t)$; his solution is as follows:

By the mean-value theorem.

$$f(x+h)=f(x)+h\,f'(x+\xi),\ 0<\xi<h; \tag{1}$$

also, by the mean-value theorem applied to $f'(\xi)$,

$$f'(x+\xi)=f'(x)+\xi f''(x+\xi_1),\ 0<\xi_1<\xi. \tag{2}$$

Therefore, comparing (1) and (2), we have

$$\xi f''(x+\xi_1)=\frac{f(x+h)-f(x)}{h}-f'(x)=\frac{f(x+h)-f(x)-hf'(x)}{h}.$$

[1] l.c., p. 307.

i.e., putting θh for ξ,

$$\theta f''(x+\xi_1)=\frac{f(x+h)-f(x)-hf'(x)}{h^2}. \tag{3}$$

But, by (2) of Art. 39, the left side of the above is equal to $\frac{1}{2}\{f''(x)+\epsilon\}$. As $f''(t)$ is continuous and finite at $t=x$, and ξ_1 tends to 0 with h,

$$f''(x+\xi_1)=f''(x)+\epsilon_2,$$

where ϵ_2 tends to 0 with h.

Therefore (3) gives

$$\theta\{f''(x)+\epsilon_2\}=\tfrac{1}{2}\{f''(x)+\epsilon\},$$

from which because of $f''(x)\neq 0$, we have

$$\theta(+0)=\tfrac{1}{2}.$$

§ 27.

41. The *third problem* to be considered is this: Prove that

$$\theta'(+0)=\frac{1}{24}\ \frac{f'''(x)}{f''(x)},$$

giving to $\theta(0)$ the value $\frac{1}{2}$ for the purpose of calculating $\theta'(+0)$.

The following solution, due to *Rothe*,[1] postulates, in addition to the single-valuedness of $\theta(h)$, (*i*) that in a finite neighbourhood

$$x\leq t<x+\tau,$$

$f(t)$, $f'(t)$, $f''(t)$ and $f'''(t)$ are all existent, finite and continuous, and (*ii*) that $f''(x)\neq 0$.

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}\Big\{f'''(x+\theta_3 h)\Big\},\ 0<\theta_3<1. \tag{1}$$

Also

$$f'(x+\xi)=f'(x)+\xi f''(x)+\frac{\xi^2}{2!}\Big\{f'''(x+\theta_4 h)\Big\}, \tag{2}$$

ξ standing as usual for θh, and $0<\theta_4<1$.

Therefore, comparing (1) and (2), and using the mean-value theorem

$$f(x+h)=f(x)+hf'(x+\xi),$$

[1] "Zum Mittelwertsatze und zur Taylorschen Formel" (*Tohoku Math. J.*, Vol. 29, 1928, pp. 145-57, specially p. 151). A less rigorous proof was given by T. Hayashi (*Science Reports of the Tohoku Imperial University*, Ser. I, Vol. XIII, 1925).

See also in this connection Terracini's paper in *Giornale di Batt.*, 1913.

we have

$$\frac{h^2}{2} f''(x)+\frac{h^3}{6}f'''(x+\theta_3 h)=h\xi f''(x)+\frac{h\xi^2}{2} f'''(x+\theta_4 h). \qquad (3)$$

Now from § 26, $\xi\equiv h\theta=h(\frac{1}{2}+a)$, where a tends to 0 with h. Therefore (3) becomes, on dividing by h^2,

$$\tfrac{1}{2}f''(x)+\frac{h}{6}f'''(x+\theta_3 h)=(\tfrac{1}{2}+a)f''(x)+\frac{h}{2}(\tfrac{1}{4}+a^2+a)f'''(x+\theta_4 h).$$

$$i.e.,\ a f''(x)=\frac{h}{24}f'''(x)+\frac{h}{6}\epsilon_4-\frac{h}{2}\epsilon_5(\tfrac{1}{4}+a^2+a)-\frac{h}{2}(a^2+a)f'''(x), \qquad (4)$$

putting

$f'''(x+\theta_3 h)=f'''(x)+\epsilon_4$, $f'''(x+\theta_4 h)=f'''(x)+\epsilon_5$, where ϵ_4, ϵ_5 tend to 0 with h.

Now $f''(x)\neq 0$; therefore, dividing both the sides of (4) by $f''(x)$,

$$a=\frac{h}{24}\ \frac{f'''(x)}{f''(x)}+E,$$

where $$E=\frac{h}{f''(x)}\left\{\frac{\epsilon_4}{6}-\tfrac{1}{2}\epsilon_5(\tfrac{1}{4}+a^2+a)-\tfrac{1}{2}(a^2+a)f'''(x)\right\}.$$

Therefore E may be neglected in comparison with

$$\frac{h}{24}\ \frac{f'''(x)}{f''(x)}$$

as $f'''(x)$ is finite by hypothesis.

Thus it is proved that

$$\lim_{h=+0}\frac{a}{h},\ i.e.,\ \lim_{h=+0}\frac{\theta-\frac{1}{2}}{h},\ i.e.,\ \theta'(+0),=\frac{1}{24}\ \frac{f'''(x)}{f''(x)}.$$

42. If $f'''(x)$ exists and is *finite*, without any regard to what may be the behaviour of $f'''(t)$ at points other than $t=x$, the above result holds, it being assumed that, in addition to the function $\theta(h)$ being single-valued,

$f(t)$, $f'(t)$, $f''(t)$ are all finite and continuous in an interval $x<t<\tau$. This follows from the result given by Dini[1] that, ϵ_n tending to o with h,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\dots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x)$$
$$+\frac{h^n}{n!}\left\{ f^{(n)}(x)+\epsilon_n \right\},$$

if $f^{(n)}(x)$ be existent and finite and the lower differential co-efficients of $f(t)$ are all finite and continuous in an interval $x\leq t\leq\tau$.

§ 28.

43. I proceed now with an account of *Ganesh Prasad's* researches[2] on the functional nature of θ; the account will be continued in the next lecture.

Prasad was the first to distinguish *clearly* between θ as a single-valued function and θ as a multiple-valued function.[3] For this purpose at the outset of his investigation, he laid down the *following fundamental theorems*,[4] taking for the sake of simplicity

$$x=0,\ f(h)=\int_0^h w(t)dt,\ \xi=\theta h$$

in the mean-value theorem

$$f(x+h)=f(x)+h\ f'(x+\theta h):—$$

Theorem Q: If, $f'(h)$ is monotone and continuous in the domain (0,1), then there is one-to-one correspondence between h and ξ, h varying in its domain (0,1) and ξ in its own domain, say $\triangle$.

Theorem R: If θ is a single-valued function of h, then the function $w(t)$ must not have an infinite number of maxima and minima in the neighbourhood of any point t in the domain $\triangle$.

[1] "Calcolo Infinitesimale," t. 1, p. 87 (foot-note). Also Genocchi-Peano's "Calcolo Differenziale," 1884, p. XIX.

[2] "On the function θ in the mean-value theorem of the Differential Calculus, (*Bulletin of the Calcutta Mathematical Society*, Vol. XX, 1928-29, pp. 155-84); "On Rolle's function θ as a multiple valued function." (*Proceedings of the Benares Mathematical Society*, Vol. X, 1929, pp. 1-10); a paper "On Rolle's function θ in the mean-value theorem for the case of a nowhere differentiable $f'(x)$," to be read before the Calcutta Mathematical Society on the 23rd March, 1930. In these lectures these papers will be quoted as the first, second and third papers of Prasad.

[3] E. R. Hedrick and, before him, Stieltjes and others knew that θ could be multiple-valued.

[4] See pp. 156-60 of Prasad's first paper.

Cor. As a *nowhere* differentiable function has an infinite number of maxima and minima in the neighbourhood of every point, it follows from the above theorem that, if θ is to be single-valued, $w(t)$ cannot be a *nowhere* differentiable function.

44. As, in his first paper, *Rothe* had postulated that θ should *be not only single-valued but also differentiable*,[1] Prasad pointed out in the following theorems that the single-valuedness of θ carried with it as a consequence its continuity but not its differentiability.

Theorem S: If θ is a single-valued function of h, then it is necessarily continuous everywhere with the possible exception of $h=0$.

Theorem T: If $\theta(h)$ is single-valued and continuous, $\theta'(h)$ need not exist for every value of h.

45. Prasad gives certain general types[2] of $f(h)$ with the *necessary and sufficient conditions* for the existence of $\theta(+0)$ for each type; the types are the following:

Type I: $w(t)=\int_0^t\{2+\sin\psi(v)\}dv,\ \psi \succ 1.$

Here $\theta(+0)$ exists or does not exist according as

$$\psi \succ \log\frac{1}{v}$$

Or

$$\psi \preceq \log\frac{1}{v}.$$

Proof.

(*a*) Let $\psi \succ \log\frac{1}{v}$. Then $w'(0)$ exists[3] and equals 2; that is, $f''(0)$ exists and is 2. Hence Dini's condition is satisfied; also, because of the integrand being always positive, $w(t)$ is monotone. Thus θ is single-valued and $\theta(+0)=\frac{1}{2}$.

(*b*) Let $\psi \preceq \log\frac{1}{v}$. Then it can be seen without difficulty that $w(t)=2t+At\cos\{\psi(t)+B\}+o(1)$, where A is a constant different from zero and B is another constant.

[1] See *l. c.*, p. 302.

[2] See pp. 172-74 of his first paper. I take this opportunity to point out an error in that paper at the end of p. 173. The conditions should be: $\psi \succ \log\frac{1}{v}$ or $\psi \preceq \log\frac{1}{v}$.

[3] See Prasad's paper, "On the differentiability of the integral function" (*Crelle's Journal*, Bd. 160, 1929); also Prasad's papers in the *Bulletin of the Cal. M. S.*, Vol. 16.

Also $f(h)=h^2+A_1h^2\cos\{\psi(h)+B_1\}$, A_1 being a constant $\neq 0$ and B_1 another constant.

Now by the mean-value theorem

$$f(h)=hw(\theta h).$$

Therefore

$\theta(+0)$ is non-existent as $\displaystyle\lim_{h\to+0}\frac{1+A_1\cos\{\psi(h)+B_1\}}{2+A\cos\{\psi(\theta h)+B\}}$ is non-existent;

because if the limit $\theta(+0)$ existed

$$\lim_{h\to+0}\frac{1+A_1\cos\{\psi(h)+B_1\}}{2+A\cos[\psi\{h\theta(+0)\}+B]} \quad \cdots \quad \cdots \quad (1)$$

would exist; which is not possible, as $\theta(+0)$ cannot be 1 as is obvious from the form of the expression (1).

Type II: $w(t)=\displaystyle\int_0^t\chi(v)\,\{2+\sin\psi(v)\}dv$, $\chi\succ 1$. $\theta(+0)$ exists or does not exist according

$$\psi \succ \log\frac{1}{v}$$

Or,

$$\psi \preceq \log\frac{1}{v}.$$[1]

Proof.

(a) Let $\dfrac{\chi}{\psi'}\prec v$. Then, denoting $\displaystyle\int_0^t\chi(v)\,dv$ by $X_1(t)$,

$$w(t)=2X_1(t)-\frac{\chi(t)}{\psi'(t)}\cos\psi(t)+o_1(t), \quad (1)$$

$$f(h)=2X_2(h)-\frac{\chi(h)}{\{\psi'(h)\}^2}\sin\psi(h)+o_2(h^2), \quad (2)$$

where $X_2(h)$ denotes $\displaystyle\int_0^h X_1(t)\,dt$.

But by the mean-value theorem

$$f(h)=h\,w(\xi).$$

[1] These are the correct conditions. In §3 of the first paper of Prasad, Examples 2 and 3 obviously violate the conditions given in Art. 14 and satisfy the correct conditions. The error seems to have crept in by some inadvertence.

Therefore, dividing both the sides of the above equation by $2X_2(h)$ and using (1) and (2), we have

$$1-\frac{\chi(h)}{2X_2(h)\{\psi'(h)\}^2}\sin\psi(h)+\frac{o_2(h^2)}{2X_2(h)}=\frac{hX_1(h\theta)}{X_2(h)}-\frac{h\chi(h\theta)}{2X_2(h)\psi'(h\theta)}$$

$$\times\cos\psi(h\theta)+\frac{ho_1(h)}{2X_2(h)}. \qquad (3)$$

Now let h tend to 0 ; then (3) gives

$$1=\operatorname*{Lim}_{h\to+0}\left\{\frac{hX_1(h\theta)}{X_2(h)}\right\};$$

the other terms in the equality singly tending to 0. But X_2 is of the same order as hX_1 ; therefore the right side gives a function of θ, say $\phi(\theta)$, whence we get $\theta(+0)$.

(*b*) The other case may be dealt with by using the methods of Arts. 10-13 of Prasad's paper in *Crelle's Journal*, Bd. 160, and it will be found that $\theta(+0)$ is non-existent.

46. *Illustrative Examples.*

Ex. 1. Let $w(t)=\int_0^t v^{-\frac{1}{2}}\left\{2+\sin\frac{1}{\sqrt{v}}\right\}dv.$

Then $\theta(+0)$ exists and equals $\frac{4}{9}$, although

$$f''(0)=\operatorname*{Lim}_{h\to+0}\frac{w(h)}{h}=\infty,\text{ and } f''(h)=h^{-\frac{1}{2}}\left((2+\sin\frac{1}{\sqrt{h}}\right).$$

Ex. 2. Let $w(t)=\int_0^t v^{-\frac{1}{2}}\left(2+\sin\frac{1}{\sqrt[3]{v}}\right)dv.$

Then $f''(0)$ is ∞ and still $\theta(+0)$ exists being $\frac{4}{9}$.

Ex. 3. Let $w(t)=\int_0^t\left(2+\sin\log_2^2\frac{1}{v}\right)dv.$

Then $\theta(+o)$ is non-existent.

§ 29

47. Prasad also gives certain general types [1] of $f(h)$ with the *necessary and sufficient conditions* for the existence of $\theta'(+o)$ for each type ; the types are the following:

Types I and II: As regards Type I, it follows from (a) of Art. 45 that $\theta(+o)=\frac{1}{2}$. Now, as $\psi \succ \log \frac{1}{v}$,

$$w(t)=2t+\int_o^t \sin \psi(v)dv=2t-\frac{\cos \psi(t)}{\psi'(t)}+o_3\ (t).$$

$$f(h)=\int_o^h w(t)\ dt=h^2-\frac{\sin \psi(h)}{\{\psi'(h)\}^2}+o_3\ (h^2).$$

But, by the mean-value theorem, $f(h)=hw\ (\xi)$.

Therefore

$$h^2-\frac{\sin \psi(h)}{\{\psi'(h)\}^2}+o_3(h^2)=h\left\{2\xi-\frac{\cos \psi(\xi)}{\psi'(\xi)}\right\}+h\ o_3\ (\xi).$$

Dividing both the sides of the above by h^2,

$$1-\frac{\sin \psi(h)}{\{h\psi'(h)\}^2}+o_3\ (1)=\left\{2\theta-\frac{\cos \psi(h\theta)}{h\psi'(h\theta)}\right\}+o_3\ (\theta).$$

Now

$$\operatorname*{Lim}_{h\to+0}\frac{\theta-\frac{1}{2}}{h}=\operatorname*{Lim}_{h\to+0}\frac{\cos\ \psi(\frac{h}{2})}{h^2\psi'(\frac{h}{2})},\ \text{as } h\ \psi'(h)\succ 1.$$

Therefore $\theta'(+o)$ exists or not according as $\psi(v)\succ\frac{1}{v}$ or not ; when $\theta'(+o)$ exists it equals 0.

Similarly, as regards Type II it follows from Art. 45 that if $\psi(v)\succ \log\frac{1}{v}$, $\theta'(+o)$ exists or not according as $\frac{\chi(v)}{X_2(v)\psi'(v)}\prec 1$ or not.

Type III: $w(t)=\int_o^t W(v)\ dv,\ W(v)=2+\int_o^v Y(u)du,$

$$Y(u)=2+\sin \psi(u),\ \psi\succ 1.$$

[1] See pages 174-77 of his first paper.

It has been proved by Prasad that $\theta'(+o)$ exists or not according as

$$\psi \succ \log \frac{1}{u}$$

Or

$$\psi \precsim \log \frac{1}{u}.$$

Type IV : $w(t)=\int_o^t W(v)dv,\ W(v)=2+\int_o^v Y(u)du,$

$$Y(u)=\chi(u)\{2+\sin\psi(u)\},\ \chi\succ 1,\ \psi\succ 1.$$

$\theta'(+o)$ exists or not according as

$$\psi \succ \log \frac{1}{u}$$

Or

$$\psi \precsim \log \frac{1}{u}.$$

48. *Illustrative Examples.*

Ex. 1. Let $w(t)=\int_o^t W(v)\,dv,\ W(v)=2+\int_o^v Y(u)du.$

$$Y(u)=u^{-\frac{1}{2}}\left\{2+\sin\frac{1}{\sqrt{u}}\right\}.$$

Then $\theta'(+o)$ exists and is ∞ ; although $f'''(o)=\infty$ and

$$f'''(h)=h^{-\frac{1}{2}}\left\{2+\sin\frac{1}{\sqrt{h}}\right\}.$$

Ex. 2. Let $w(t)=\int_o^t\left\{2+\sin\frac{1}{v}\right\}dv.$

Then $\theta(+o)$ exists and equals $\frac{1}{2}$ but $\theta'(+o)$ is non-existent.

§ 30.

49. I will conclude to-day's lecture by giving from Prasad's work functions θ each single-valued and each non-differentiable at the points of an everywhere dense set.[1]

[1] See pp. 178-80 of Prasad's first paper.

Functions θ non-differentiable at the rational points: *Type A.*

(α) Let $\phi(z)$ stand for $10z^2+z^2\cos\log\frac{1}{z^2}$. Then $\phi'(z)$ will be a monotone, increasing and continuous function for every value of z, positive or negative.

$$\text{For, } \phi'(z)=20z+2\sqrt{2}z\cos\left(\log\frac{1}{z^2}-\frac{\pi}{4}\right).$$

$$\phi''(z)=20+2\sqrt{10}\cos\left(\log\frac{1}{z^2}-\frac{\pi}{4}-\tan^{-1}\tfrac{1}{2}\right).$$

Therefore it is proved that, for every value of z, $\phi''(z)>0$ and therefore $\phi'(z)$ is monotone and increasing; that it is also continuous is obvious.

(β) Thus, if $\{\omega_n\}$ be the set of rational points in the interval $(0,1)$,

$$f(h)=\sum_{n=1}^{\infty}\frac{\phi(h-\omega_n)-\phi(\omega_n)}{n^2}$$

will give $f(0)=0$ and $f'(h)=\sum_{n=1}^{\infty}\frac{\phi'(h-\omega_n)}{n^2}$ which will be monotone, increasing, and continuous for every value of h in $(0, 1)$, because such is $\phi'(h-w_n)$.

Therefore, by Theorem Q, there is one-to-one correspondence between h and ξ as each moves in its domain of variability. If, then, ξ is a rational number, say ω_m, there will be a corresponding value of h, say h_m. Now, by the mean-value theorem

$$f(h)=hw(\xi).$$

Since $f'(h)$ exists for every value of h,

$$f'(h_m)=\left[\frac{d}{dh}\{hw(\xi)\}\right]_{h=h_m}=w(\xi_m)+h_m\left\{\frac{d}{dh}w(\xi)\right\}_{h=h_m}.$$

Thus it is proved that, for $h=h_m>0$,

$$\frac{d}{dh}\{w(\xi)\}\text{ exists and equals }\frac{f'(h_m)-w(\xi_m)}{h_m}.$$

But $\left(\frac{d\xi}{dh}\right)_{h=h_m}$ must not exist ; for, if it were to exist, $w'(\xi_m)$ would

exist and, in fact, equal $\dfrac{\left[\dfrac{d}{dh}\{w(\xi)\}\right]_{h=h_m}}{\left(\dfrac{d\xi}{dh}\right)_{h=h_m}}$, which will be absurd as

$w'(\xi_m)$ is non-existent.

If 0 is included in $\{w_n\}$, it can be proved that $\theta(+0)$ is non-existent because of the inclusion of the term $h^2 \cos \log \frac{1}{h^2}$ in $f(h)$.

50. *A Broden's function taken for* $w(t)$. *Type B.*—The three non-differentiable functions given by Broden[1] in *Crelle's Journal*, Bd. 118, are all monotone, increasing and continuous. Therefore, if $f(h)=\int_0^h w(t)dt$,

where $w(t)$ is one of such functions, there will be one-to-one correspondence between h and ξ as each moves in its domain of variability. Now each of the functions is non-differentiable at the points of an everywhere dense set, enumerable or unenumerable. Thus, there are, corresponding to these functions of Broden, functions θ each of which is non-differentiable at an everywhere dense set, enumerable or unenumerable.

[1] Denoting in each case the function by $f(x)$ for $(0, 1)$ where the ends are primary points, Broden says the following about the three functions : (1) "The function is continuous and throughout increases with x; the derivatives $f'+(x)$ and $f'-(x)$ are everywhere definite, finite and different from 0; they are also equal to one another [and to $f'(x)$] with the exception of an enumerable and everywhere dense set of x-values (*viz.*, those corresponding to the primary points). (See pp. 27 and 28 of Broden's paper.) (2) "$f(x)$ is continuous and throughout increases with x; for a certain enumerable and everywhere dense set of x-values (*viz.*, those corresponding to the primary points) $f'_-(x)=0$, and $f'_+(x)$ has a definite value>0; for an unenumerable and everywhere dense set of x-values is $f'_-(x)=f'_+(x)=0$, for another such set is $f'_-(x)$ non-existent but $f'_+(x)$ existent and >0. (See p. 37, *l.c.*) (3) "$f(x)$ is continuous and increases throughout with x; for an everywhere dense and enumerable set of x-values is the regressive derivative $f'_-(x)=0$ but the progressive $f'_+(x)=\infty$, for an everywhere dense and unenumerable set of x-values is $f'_-(x)=f'_+(x)>0$; for a second such set is $f'_-(x)=f'_+(x)=0$; for a third $f'_-(x)=f'_+(x)=\infty$; for a fourth set of the same kind are $f'_-(x)$ and $f'_+(x)$ both non-existent ; for a fifth is $f'_-(x)=0$ but $f'_+(x)$ non-existent." (See pp. 46 and 47, *l.c.*) The first set in the above includes the primary points.

An interesting question arises: As in the case of each function of Broden, 0 is a primary point and, therefore, a point of non-differentiability for $w(t)$, $w'(0)$ is non-existent; what can be said about the existence of $\theta(+0)$ and $\theta'(+0)$?

The answer to this question may be attempted as follows:

(*a*) For each of the first two functions $w'(+0)$, which has been designated $f''(0)$ in the preceding pages, exists, is finite and greater than 0; hence in the case of each by Dini's result, $\theta(+0)$ exists and equals $\frac{1}{2}$. In the case of the third function of Broden, $w'(+0)$ is ∞, and it is difficult to prove that $\theta(+0)$ exists.

(*b*) As regards $\theta'(+0)$, its existence is unlikely; for, in any neighbourhood of 0 ever so small the points where $w'(h)$ is non-differentiable are everywhere dense.

FIFTH LECTURE

THE FUNCTIONAL NATURE OF θ (CONTINUED).

§ 31

51. Just as the last lecture was devoted to the study of the functional nature of θ as a *single-valued* function of h; so in to-day's lecture will be discussed chiefly the functional nature of θ as a *multiple-valued* function of h. Before giving an account of Prasad's researches relating to θ as a multiple-valued function, I reproduce on account of their historical interest the following passages from the first important contribution to the subject, *viz.*, Hedrick's paper,[1] "On a function which occurs in the law of the mean":—

(*a*) "The formula

$$f(x+h)-f(x)=h.\,f'(x+\theta h)=h\,f'(x+\xi),\ 0<\theta<1,$$

is known to every student of mathematics under the title 'the law of the mean' or a synonym. It is usually proved that the formula is correct if $f(x)$ is defined in an interval $a\leq x\leq b$ and if the derivative $f'(x)$ exists in the continuum $a<x<b$.

The quantity ξ which occurs in the formula is evidently a function of the two independent variables x and h and is defined for all values of x and h for which x and $x+h$ both lie in the interval in which $f'(x)$ exists. The purpose of the present paper is to discuss the properties of this quantity ξ, and we shall write

$$\xi=\xi\ (x,\ h)$$

whenever it is desired to emphasize its functional character.

In studying the function $\xi(x,\ h)$ it occurs to one immediately that if $\xi\ (x,\ h)$ is continuous, the derivative $f'(x)$ is surely continuous (see p. 183). A slight inspection would tend to convince one that the converse is

[1] *Annals of Mathematics*, Vol. 7, July, 1906, pp. 177-192; specially p. 177.

"This paper is based partially upon a paper entitled 'On the Law of the Mean' read by the writer at the summer meeting, 1903, of the American Mathematical Society, and partially, on another paper entitled 'The function $\xi\ (h)$ in the Law of the Mean' read by the writer at the April meeting, 1906, of the Chicago Section of the Society" (see footnote to p. 177 of the *Annals*).

true—at least in a limited sense—*i.e.*, that if $f'(x)$ is continuous, $\xi(x, h)$ is a continuous function of h when x is constant. This conclusion is, however, erroneous. It will be shown that $\xi(x, h)$ may be utterly discontinuous,[1] in the sense that ξ fails to take on values which lie between values which it does take on, even when $f'(x)$ exists and is continuous everywhere (see p. 190).

The fallacies which depend upon the utter discontinuity of ξ are known, but not well-known. While pointing out their dangers, it may be well to remark that certain proofs may be conducted without error along the same general lines as these fallacious arguments, if the function ξ be fully understood.

Incidentally it will become evident that the law of the mean may hold in its present form in case the derivative $f'(x)$ does not even exist at certain points,[2] even when we permit x and $(x+h)$ to assume any values whatever; and generalized statements of the law hold even in more tenuous cases."

(*b*) "We shall now assume, however, that $f'(x)$ exists in the ordinary sense. The number ξ depends upon the choice of both x and h, and may have several values, or even an infinite number, for one pair of values of x and h.[3] If x is fixed, ξ has at least one value for every value of h and that value is numerically less than h."

(*c*)[4] "Let us now consider a function $f(x)$ whose derivative exists and is continuous throughout some interval $a \leq x \leq b$. We know that a sufficient condition for the continuity of $f'(x)$ is that $\xi(x, h)$ be a continuous function of h when x is a constant, or even that $|\xi|$ assume all values less than λ for some value of $|h|$ less than ρ. Conversely, *can we infer that if $f'(x)$ is continuous, it will follow that $\xi(x, h)$ is a continuous function of h when x is constant? Or even that $|\xi|$ assumes all values less than λ for some value of $|h|$ less than ρ?* Such a conclusion seems to be obvious at first sight, but it is untrue.

Let us consider, for example, the function

$$y=f(x)=x^3\left(1+\sin\frac{1}{x}\right) \text{ if } x \neq 0.$$

$$y=f(x)=0 \qquad \text{if } x=0.$$

[1] The example of p. 189 is reproduced below under (*c*).

[2] P. 181. The case in which $f(0)=0$ and $f(x)=\sin 1/x$ illustrates the statement.

[3] P. 182. As stated in the first lecture, it was known to Cambridge mathematicians as early as 1891 that θ may be multiple-valued. Very probably, Stieltjes was also aware of this property of θ as early as 1882. See *Oeuvres*, t. 1, pp. 67-72.

[4] P. 189.

The derivative is

$$y'=f'(x)=3x^2\left(1+\sin\frac{1}{x}\right)-x\cos\frac{1}{x} \quad \text{if } x\neq 0,$$
$$y'=f'(x)=0 \qquad\qquad \text{if } x=0,$$

which is continuous for all values of x. Nevertheless there are values of $\xi\,(o,\,h)$ which are not taken on for any value of x whatever and which lie as close to zero as we please.

For it is obvious[1] that

$$f'(x)=0 \text{ when } x=\frac{2}{(4n+1)\pi}$$

and that

$$f(x)=0 \text{ when } x=\frac{2}{(4n+1)\pi}\ ;\ f(x)>0 \text{ when } x\neq\frac{2}{(4n+1)\pi}.$$

Now $f'(x)\neq 0$, since $f'(x)\neq$ constant. Moreover $f'(x)\geq 0$ is surely false since $f(0)$ is not a minimum to the right. Hence $f'(x)<0$ for some value of x, say x_n, which lies between $\frac{2}{(4n+1)\pi}$ and $\frac{2}{(4n+5)\pi}$. It follows that ξ never takes on the value x_n, since in the formula

$$f(h)-f(0)=h\,f'(\xi),\ 0<\xi<h,$$

the right-hand side is never negative.

We may therefore conclude that

XIX. "*It is entirely possible that $f'(x)$ exists and is continuous throughout an interval about a point $x=k$, and yet that the assemblage of values which the function $\xi\,(k,\,h)$ never assumes has the power of the continuum.* For since $f'(x)$ is continuous, it assumes negative values (say) for all values of x in the neighbourhood of any point when it is negative."

§ 32

52. The following general theorem and definition lie at the foundation of Prasad's researches,[2] where for the sake of simplicity x is taken to be 0, $f(0)=0$, so that the mean-value theorem becomes $f(h)=h\,f'(\theta h)$, $0<\theta<1$:—

[1] In the next 8 lines there are a number of misprints; but the conclusion given in the last sentence of those lines is easily arrived at by correcting the misprints.

Line 2 and Line 4, for "$(4n+1)$" read "$(4n+3)$".
Line 5 , for "since $f'(x)$," read "since $f(x)$."
Line 7 , for "$(4n+1)$" read "$(4n+3)$"
, for "$(4n+5)$" read, "$(4n+7)$"

[2] See his second and third papers.

(a) *Theorem*: The roots of the equation in θ,

$$f(h)=hf'(\theta h), \qquad \ldots \quad (1)$$

form either a finite set or an infinite but enumerable and nowhere dense set.

Proof:

Let $F(t)$ denote $f(t)-\frac{f(h)}{h}t$. Then, since according to a known theorem of Schoenflies [1] the maxima and minima of every continuous function form a finite or enumerable set, the values of t for which

$$F'(t),\ i.e.,\ f'(t)-\frac{f(h)}{h}=0$$

form such a set. Thus, as $h \neq 0$, the values of θ satisfying (1) form such a set.

That, when $f'(t)$ is continuous, this set cannot be everywhere dense is obvious.

For, if the set were everywhere dense any number could be looked upon as a limiting point of a sequence of θ's satisfying (1) and would therefore itself satisfy (1) as $f'(t)$ is continuous. Therefore any number in a continuum will in that case satisfy (1), which will be absurd.

(b) *Definition.* The values of θ satisfying (1) may be arranged in order of their magnitudes and denoted by $\theta_1, \theta_2, \theta_3, \ldots\ldots\theta_m, \ldots\ldots$; θ_1 being the greatest of all the values. Prasad calls $\theta_1(h)$ the *principal value* of θ for a given h; obviously $\theta_1(h)$ is a single-valued function of h.

§ 33

53. The following simple examples illustrate the theorem and definition of § 32.

Example I. Take

$$f(t)=\frac{t}{\sqrt{2}}\cos\left(\frac{1}{2}\log\frac{1}{t^2}+\frac{\pi}{4}\right).$$

Then $f'(t)$ exists for every value of t other than 0 and equals

$$\cos\left(\frac{1}{2}\log\frac{1}{t^2}\right).$$

Therefore the mean-value theorem (1) of § 33 becomes

$$f(h)=h\cos\left(\frac{1}{2}\log\frac{1}{\theta^2h^2}\right).$$

[1] See p. 158 of Schoenflies' "Die Entwickelung der Lehre von den Punktmannigfaltigkeiten," 1900.

Therefore θ is given by the transcendental equation

$$\cos\left(\frac{1}{2}\log\frac{1}{\theta^2h^2}\right)=\frac{1}{\sqrt{2}}\cos\left(\frac{1}{2}\log\frac{1}{h^2}+\frac{\pi}{4}\right).$$

Taking $\alpha(h)$ to be the numerically least angle whose cosine equals the right side of the above equation, and denoting by $N(h)$ a suitable integer dependent on h, we have

$$\frac{1}{2}\log\frac{1}{\theta^2h^2}=2N(h)\pi\pm\alpha,$$

i.e., taking h to be positive,

$$\theta=\frac{1}{h}e^{-2N\pi\mp\alpha},$$

where the integer N is so chosen that θ lies between 0 and 1.

If N_1 is any such integer for a given h, it is obvious that N_1+1, N_1+2,...... are all such integers. For instance, if h is $e^{-2m\pi}$, m being an integer,

$$\frac{1}{\sqrt{2}}\cos\left(\frac{1}{2}\log\frac{1}{h^2}+\frac{\pi}{4}\right)=\frac{1}{2}=\cos\frac{\pi}{3};$$

thus

$$\alpha=\frac{\pi}{3}$$

and

$$\theta=e^{2(m-N)\pi}\mp\frac{\pi}{3}.$$

The principal value θ_1 is $e^{-\frac{\pi}{3}}$; the other values in order of magnitude are

$$e^{-2\pi+\frac{\pi}{3}},\quad e^{-2\pi-\frac{\pi}{3}},\quad e^{-4\pi+\frac{\pi}{3}},\quad e^{-4\pi-\frac{\pi}{3}},\quad \dots\quad \dots$$

Example II. Take

$$f(t)=\int_0^t\cos\frac{1}{v}\,dv.$$

Then $f'(t)$ exists for every value of t including 0.

Therefore the mean-value theorem (1) of § 32 becomes

$$f(h)=h\,f'(\theta h)=h\cos\frac{1}{\theta h}.$$

Therefore θ is given by the transcendental equation.

$$\cos\frac{1}{\theta h}=\frac{f(h)}{h},$$

in which the right side behaves as $-h\sin\frac{1}{h}$ for small values of h. Taking $\alpha(h)$ to be the numerically least angle whose cosine equals the right side of the above equation, we have

$$\frac{1}{\theta h}=2\mathrm{N}(h)\pi\pm\alpha,$$

i.e.,
$$\theta=\frac{1}{h(2\mathrm{N}\pi\pm\alpha)},$$

where N is any integer so chosen that θ lies between 0 and 1.

If N_1 is such an integer, then obviously N_1+1, N_1+2, ... are all such integers.

For instance, if $h=\dfrac{1}{2m\pi+\frac{\pi}{2}}$, where m is any large integer,

$$\frac{f(h)}{h}\text{ is nearly}-\frac{1}{2m\pi+\frac{\pi}{2}};$$

thus
$$\alpha\text{ is nearly }\frac{\pi}{2}+\frac{1}{2m\pi+\frac{\pi}{2}}$$

and
$$\theta=\frac{2m\pi+\frac{\pi}{2}}{2\mathrm{N}\pi\pm\left(\frac{\pi}{2}+\frac{1}{2m\pi+\frac{\pi}{2}}\right)}\text{ nearly.}$$

The principal value θ_1 is $\dfrac{2m\pi+\frac{\pi}{2}}{2m\pi+\frac{\pi}{2}+\dfrac{1}{2m\pi+\frac{\pi}{2}}}$ nearly, the other values,

in order of magnitude, are nearly

$$\frac{2m\pi+\frac{\pi}{2}}{2(m+1)\pi-\left(\frac{\pi}{2}+\frac{1}{2m\pi+\frac{\pi}{2}}\right)},\ \frac{2m\pi+\frac{\pi}{2}}{2(m+1)\pi+\frac{\pi}{2}+\frac{1}{2m\pi+\frac{\pi}{2}}},$$

$$\frac{2m\pi+\frac{\pi}{2}}{2(m+2)\pi-\left(\frac{\pi}{2}+\frac{1}{2m\pi+\frac{\pi}{2}}\right)},\ \frac{2m\pi+\frac{\pi}{2}}{2(m+2)\pi+\frac{\pi}{2}+\frac{1}{2m\pi+\frac{\pi}{2}}}\cdots\cdot$$

§34

54. That the principal value of θ is not necessarily continuous at $h=0$ is proved by the non-existence of $\theta_1(+0)$ in the case of Example I of § 33.

Proof: Let h tend to 0 by taking the values of the form $e^{-2m\pi}$, where m is positive and integral; then, as shown in connection with the study of Example I in § 33, $\theta_1(h)$ is always $e^{-\frac{\pi}{3}}$. Thus for this mode of approach to 0, $\theta_1(h)$ tends to $e^{-\frac{\pi}{3}}$.

If h is of the form $e^{-\left(2m\pi+\frac{\pi}{2}\right)}$, where m is positive and integral, then

$$\frac{1}{\sqrt{2}}\cos\left(\frac{1}{2}\log\frac{1}{h^2}+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}\left(\cos 2m\pi+\frac{\pi}{2}+\frac{\pi}{4}\right)=-\tfrac{1}{2}.$$

Therefore $\quad a=\frac{2\pi}{3}$ and $\theta_1(h)$ is equal to $e^{\frac{\pi}{2}-\frac{2\pi}{3}}$ *i.e.*, $e^{-\frac{\pi}{6}}$.

Thus for this mode of approach to 0 for h, *viz.*, by taking values of the form

$$e^{-\left(2m\pi+\frac{\pi}{2}\right)}$$

$\theta_1(h)$ tends to $e^{-\frac{\pi}{6}}$.

Therefore $\theta_1(+0)$ does not exist.

§ 35.

55. That the principal value of θ, if continuous at $h=0$, is not necessarily differentiable there, is proved by considering Example II of § 35.

In this case in the neighbourhood of $h=0$, $f(h)$ behaves as

$$-h^2 \sin\frac{1}{h}$$

Therefore, as shown in § 33,

$$\theta(h)=\frac{1}{h\,(2N\pi \pm a)}$$

and $\theta_1(h)$ tends to 1 as h tends to 0. Hence, *assuming* that $\theta_1(0)$ is 1, it is proved that $\theta_1(h)$ is continuous at $h=0$.

56. I proceed now to prove that $\theta'_1(+0)$ is non-existent.

Proof:

(a) Consider

$$\frac{\theta_1(h)-1}{h}$$

for the values of h of the sequence

$$\left\{\frac{1}{2m\pi+\frac{\pi}{2}}\right\}.$$

Then we have, in the equation

$$\theta=\frac{1}{h(2N\pi\pm a)} \text{ of § 33,}$$

a of the form [1]

$$\frac{1}{2m\pi+\frac{\pi}{2}}+\frac{\pi}{2}+ \text{ higher powers of } \frac{1}{m} \text{ than the first.}$$

$$\text{Therefore} \qquad \theta_1(h)=\frac{2m\pi+\frac{\pi}{2}}{2m\pi+a}$$

and

$$\frac{\theta_1(h)-1}{h}=\left(2m\pi+\frac{\pi}{2}\right)\left\{-\frac{1}{\left(2m\pi+\frac{\pi}{2}\right)^2}+ \text{ higher powers of } \frac{1}{m} \text{ than the first.}\right\}$$

[1] Cos a is nearly equal to $-h\sin\frac{1}{h}$, *i.e.*, $-\frac{1}{2m\pi+\frac{\pi}{2}}$.

Hence for the sequence under consideration

$$\underset{h=+0}{\text{Lim}} \quad \frac{\theta_1(h)-1}{h}=0.$$

(*b*) Again consider

$$\frac{\theta_1(h)-1}{h}$$

for the values of the sequence

$$\left\{\frac{1}{(2m+1)\pi}\right\}.$$

Then we have α of the form

$\frac{\pi}{2}$ + higher powers of $\frac{1}{m}$ than the first.

Therefore

$$\theta_1(h)=\frac{(2m+1)\pi}{2m\pi+2\pi+\alpha},$$

and

$$\frac{\theta_1(h)-1}{h}=(2m+1)\pi\left\{\frac{-\frac{3\pi}{2}}{(2m+1)\pi+\frac{3\pi}{2}}+\text{higher power of } \frac{1}{m} \text{ than the first}\right\}.$$

Hence for the sequence under consideration

$$\underset{h=+0}{\text{Lim}} \quad \frac{\theta_1(h)-1}{h}=-\frac{3\pi}{2}$$

which is different from the limit obtained under (*a*).

Therefore

$$\underset{h=+0}{\text{Lim}} \quad \frac{\theta_1(h)-1}{h}$$

is non-existent, and, consequently, $\theta_1(h)$ has no differential co-efficient at $h=0$.

§ 36.

57. I proceed now to consider the statement made in passage (c) quoted in § 32 from Hedrick's paper that, even if $f'(t)$ is continuous, $\xi(x, h)$ need not be continuous in h for a given x. The statement has been made because of a confusion due to Hedrick's not distinguishing between the different values of ξ for a given h.

If we stick to any particular θ, say θ_1, then the corresponding ξ, say ξ_1, is a single-valued function of h; and the reasoning of Art. 1 of my first paper [1] is quite applicable to ξ_1.

Thus $\xi_1(h)$ can have no point of discontinuity for any value of h other than 0.

58. To make my criticism of Hedrick's statement clearer, let us examine his example,

$$f(t)=t^2\left(1+\sin\frac{1}{t}\right),\ t \neq 0,$$

$$f(0)=0.$$

Assume that $\bar{h}>0$ is a point of discontinuity of the single-valued function $\xi_1(h)$. Then $\bar{h}$ cannot be a point of discontinuity of the first kind for ξ_1. For, if it were, for any sequence $\{h_n\}$ tending to $\bar{h}$ the corresponding sequence $\{\xi_{1,n}\}$ does not tend to $\bar{\xi}_1$ but to $\bar{\xi}_1'$ *different* from $\bar{\xi}_1$. Thus

$$\frac{f(\bar{h})}{\bar{h}}=f'(\bar{\xi}_1),\ \frac{f(\bar{h})}{\bar{h}}=f'(\bar{\xi}_1');$$

and so for the same value $\bar{h}$ of h there are two different values, $\bar{\xi}_1$ and $\bar{\xi}_1'$ of ξ_1. Therefore ξ_1 is not single-valued which is absurd.

Again $\bar{h}$ cannot be a point of discontinuity of the second kind for ξ_1. For, if it were, there must be a sequence $\{h_n\}$ tending to $\bar{h}$, the sequence $\{\xi_{1,n}\}$ corresponding to which does not tend to any limit. Therefore, for a neighbourhood of $\bar{h}$, as small as we please, there must be values of ξ_1, say $\xi_{1,n_1}, \xi_{1,n_2}, \ldots \xi_{1,n_r}, \ldots$ differing from one another by more than any suitably chosen quantity $\delta>0$. But $f'(\xi_{1,n_1}), f'(\xi_{1,n_2}), \ldots$ are different from one another only by a quantity as small as we please, because of the

[1] See the second foot-note on p. 56.

continuity of $\frac{f(h)}{h}$ at h. Therefore

$$3\,\xi_{1,n_1}^2\left(1+\sin\frac{1}{\xi_{1,n_1}}\right)-\xi_{1,n_1}\cos\frac{1}{\xi_{1,n_1}},\ 3\,\xi_{1,n_2}^2$$
$$\times\left(1+\sin\frac{1}{\xi_{1,n_2}}\right)-\xi_{1,n_2}\cos\frac{1}{\xi_{1,n_2}},\ldots$$

...differ from one another by a quantity as small as we please, which is not possible if ξ_{1,n_1}, ξ_{1,n_2},...differ from one another by more than δ.

§ 37.

59. I proceed now to consider at some length Prasad's study of the interesting case of Rolle's function when $f'(t)$ is nowhere differentiable. Take

$$f(t)=\int_0^t \omega(v)\,dv$$

where $\omega(v)$ is a continuous, but nowhere differentiable, function, say Weierstrass's function

$$\sum_1^\infty \frac{\cos(13^n v)}{2^n}.$$

Then the principal value $\theta_1(h)$ is a single-valued and continuous but *nowhere differentiable function.*[1]

Proof:

(a) That $\theta_1(h)$ is single-valued is obvious; that it is continuous for every value of $h>0$ is clear from the facts, *viz.*, (i) that in the equation

$$\frac{f(h)}{h}=\omega(\theta_1 h),$$

the left side being continuous, $\omega(\theta_1 h)$ must be also continuous and (ii) that ω being a continuous function of its argument the argument $\theta_1 h$ and, consequently, θ_1 must be continuous.

(b) The non-differentiability of θ_1 follows from the fact that in the equation

[1] See Prasad's second paper.

$$f(h)=h\omega(\xi),$$

where ξ stands for $\theta_1 h$, the right side is differentiable for every value of h. Therefore

$$f'(h)=\omega(\xi)+h\frac{d}{dh}\{\omega(\xi)\}.$$

Thus $\frac{d}{dh}\{\omega(\xi)\}$ exists, being in fact

$$\frac{f'(h)-\omega(\xi)}{h}.$$

But $\frac{d\xi}{dh}$ cannot exist; if it existed, $\omega'(\xi)$ would exist being equal to $\frac{f'(h)-\omega(\xi)}{h} \div \frac{d\xi}{dh}$. Therefore, as $\omega'(\xi)$ is non-existent, it is proved that ξ and, consequently, θ_1 are non-differentiable.

§ 38.

60. The next step in the study of nowhere differentiable θ_1 is to determine the enumerable set $\{\theta_m\}$ corresponding to any given value of h. Prasad has actually given a formula for θ_m; for the value $h=\frac{\pi}{13}$, the formula is this:[1] Whatever positive integer k may be, there is a value of θ between

$$\frac{1}{2}+\frac{1}{13^k} \text{ and } \frac{1}{2}+\frac{\frac{3}{2}}{13^k},$$

and another value between

$$\frac{1}{2}-\frac{1}{13^k} \text{ and } \frac{1}{2}-\frac{\frac{3}{2}}{13^k}.$$

[1] See his 3rd paper in which the subject is *carefully* treated. The values of θ satisfy $\frac{f(h)}{h}=\omega\left(\theta\cdot\frac{\pi}{13}\right)$. But $f(h)=0$. Therefore θ is given by $\sum_{n=1}^{\infty}\cos\frac{\pi(\theta\cdot 13^{n-1})}{2^n}=0$, *i.e.*, by the zeroes of the function $\sum_{n=0}^{\infty}\frac{\cos(\pi\theta.13^n)}{2^n}\equiv W(\theta)$.

In Prasad's paper, "On the zeroes of Weierstrass's non-differentiable function (*Proceedings of the Benares M. S.*, Vol. XI) an inaccuracy has crept in; in §2 of that paper, for "the zeroes" read " zeroes."

so that one value of θ is $\frac{1}{2}$ and the other values are given (1) by the expression

$$\frac{1}{2} \pm \frac{1}{13^4}\left(1+\frac{1}{2}\lambda_4\right),$$

λ_4 being a number between 0 and 1 which can be approximated to as closely as possible $\left[\text{it is easily proved, for example, that } \lambda_4 \text{ lies between } \frac{4}{13} \text{ and } \frac{6}{13}\right]$, and (2) by the expression $\frac{1}{2} \pm \frac{v}{13^4}$, where v is a suitable number between 3 and $3\frac{1}{2}$, or $3\frac{1}{2}$ and 4, or 5 and $5\frac{1}{2}$, or $5\frac{1}{2}$ and 6.

61. In order to answer the question: How does $\theta_m(h)$ behave as h varies?, first consider $\theta_m(h)$ when $h=\text{N}.\frac{\pi}{13}$, N being an integer. (*a*) It is obvious that the value of θ_m in this case is $\frac{1}{\text{N}}$th of the value of θ_m corresponding to the case of $h=\frac{\pi}{13}$. Thus as h increases from $\frac{\pi}{13}$, each θ decreases. (*b*) To consider the behaviour of $\theta_m(h)$ as h decreases from $\frac{\pi}{13}$, put successively $h=\frac{1}{2}\cdot\frac{\pi}{13}$, $h=\frac{\pi}{13^2}$. Then it is easily seen that in the former case there is a value of θ between $1-\frac{3}{13}$ and $1-\frac{3}{13^2}$, and in the latter case a value between 1 and $\frac{12}{13}$. We may, therefore, safely conclude that as h decreases towards 0 any particular $\theta_m(h)$ tends to 1, *i.e.*, $\theta_m(+0)=1$.

§ 39.

62. The next question to be considered is this: Admitted that $\theta'_m(h)$ is non-existent for $h>0$, does $\theta'_m(+0)$ exist or not, $\theta_m(0)$ being assumed to be 1?

This is a question of great difficulty and can be only briefly considered here. For a fairly detailed consideration, see Prasad's third paper.

There are non-differentiable functions which have each a progressive differential co-efficient at $h=0$; there are others for which this is not true. So, there are all sorts of possibilities for the behaviour of $\frac{\theta_m(h)-1}{h}$ as h tends to 0.

§ 40.

63. I will conclude this lecture by answering the following question: Is it true that, corresponding to every prescribed function $\theta(h)$, in case $\theta(h)$ is single-valued, or to every prescribed set $\{\theta_m(h)\}$, in case $\theta(h)$ is multiple-valued, there exists a function $f(h)$ for which the mean-value theorem,

$$viz., \quad f(h)=hf'(h\theta)$$

or

$$f(h)=hf'(h\theta_m)$$

holds, and if such a function $f(h)$ does not always exist what conditions must be satisfied by the prescribed function θ or the prescribed set $\{\theta_m(h)\}$ in order that $f(h)$ may exist?

Let us consider the question first for $\theta(h)$ as a single-valued function.

(*a*) If we *assume* that $\theta(h)$ is expansible in the form $\theta(h)=a_0+a_1h+a_2h^2+a_3h^3+\ldots$to infinity and also confine our search for $f(h)$ *to only such functions* as are expansible in the form

$$f(h)=A_0+A_1h+A_2h^2+A_3h^3+\ldots\text{to infinity};$$

then (taking further, without loss of generality A_0 and A_1 to be zero) we have by the mean-value theorem

$$f(h)=hf'(\theta h),$$

$$\sum_2^{\infty} A_nh^n=h\left\{\sum_2^{\infty} nA_n\,(h\theta)^{n-1}\right\}.$$

By equating the co-efficients of like powers of h, we get the values of the A's, *provided that a_0 has its value restricted to numbers of the form* $\left(\frac{1}{m}\right)^{\frac{1}{m-1}}$. Thus, if $A_2\neq 0$, a_0 *must* be $\frac{1}{2}$; if $A_2=0$, but $A_3\neq 0$, a_0 *must* be $\left(\frac{1}{3}\right)^{\frac{1}{2}}$; generally, if

$$A_2=A_3=\ldots=A_{m-1}=0 \text{ but } A_m\neq 0,$$

then a_0 *must* be $\left(\frac{1}{m}\right)^{\frac{1}{m-1}}$. With this important condition the A's are determinate and expressible in terms of the a's; *e.g.*, if $A_2\neq 0$,

$$A_3=8a_1A_2,\ A_4=4(a_2+12a^2{}_1)A_2,\ A_5=2A_2\{a_3+12a^3{}_1+ 24a_0a_1a_2+24a^2{}_0a_1a_2+24\times 12a_0{}^2a_1{}^3\}\times \tfrac{16}{11}.$$

(*b*) The general restrictions on the properties of the single-valued $\theta(h)$ are (*i*) that, for $h>0$, it must be continuous and (*ii*) that it *must not* be everywhere non-differentiable.

It is thus clear from (*a*) and (*b*) that, in order that $f(h)$ may exist corresponding to $\theta(h)$, in addition to being between 0 and 1 for all values of h, θ must fulfil certain restrictions and cannot be arbitrary. The following examples will illustrate this remark:

Ex. 1. If θ is constant it can have values only of the form $\left(\frac{1}{m}\right)^{m-1}$ when m is a positive integer.

Ex. 2-4. There is no $f(h)$ corresponding to $\theta=\frac{1}{3}+\frac{h}{2}$. For $\theta=\frac{1}{2}+\frac{h}{3}$, $f(h)=A_2\left\{h^2+\frac{8}{3}h^3+\frac{16}{3}h^4+\frac{448\times 2}{99}h^5+\ldots\right\}$, A_2 being any constant different from 0.

$$\text{For } \theta=\frac{1}{\sqrt{3}}+\frac{h}{3},\ f(h)=A_3\left\{h^3+\frac{6}{3\sqrt{3}-4}h^4+\ldots\ldots\right\}.$$

A_3 being any constant different from 0.

Ex. 5. If $\theta=\frac{1}{2}-\frac{1}{8}h\cos\frac{2}{h}$ for small values of h; then for such values $f(h)=h^2-h^4\sin\frac{1}{h}$.

Ex. 6. If $\theta^{\frac{1}{2}}=\frac{2}{3}-3h^{\frac{1}{3}}\left(\frac{2}{3}\right)^{\frac{5}{3}}\cos\frac{1}{\left(\frac{2}{3}\right)^{\frac{2}{3}}h^{\frac{1}{3}}}$ for small values of h, then for such values $f(h)=\frac{2}{3}h^{\frac{3}{2}}-9h^{\frac{13}{6}}\sin\frac{1}{h^{\frac{1}{3}}}$.

Ex. 7. If

$$\theta=\left[1+\frac{1}{\sqrt{5}}\cos\left\{\log\frac{1}{h}\ \tan^{-1}\frac{1}{2}\right\}\right]\Big/\left[2+\cos\left\{\left(\log\frac{1}{h}+\frac{1}{\theta}\right)\right\}\right],$$

then $f(h)$ will be of the form $h^2+Ch^2\cos\left\{\log\frac{1}{h}+D\right\}$, C and D being suitable constants, and in fact $C=\frac{1}{\sqrt{5}}$, $D=\tan^{-1}\frac{1}{2}$.

64. Let us consider now the case in which $\theta(h)$ is multiple-valued.

(*a*) If $\theta(h)$ has *not more than* a *finite* number values, the procedure to be adopted may be illustrated by the following examples:

Ex. 1. If ξ, *i.e.*, $h\,\theta$, is $\dfrac{h}{2}$ when h is in $(0, \frac{1}{2})$, and is for values of h in $(\frac{1}{2}, 1)$ equal to

$$\frac{h-\frac{1}{2}h^2-\frac{1}{4}}{h} \text{ or } \frac{\frac{1}{2}h^2+\frac{1}{4}}{h}$$

according as ξ lies between 0 and $\frac{1}{2}$ or between $\frac{1}{2}$ and 1; find $f(h)$ on the *assumption* that it is a power series in h.

Let $f(h)=\displaystyle\int_0^h w(t)dt$, $0<h\leq 1$, we have to find $w(t)$. It is obvious that $w(t)$ is continuous.

First consider the case of h lying in $(0, \frac{1}{2})$. As there is only one value of ξ, $w(t)$ is monotone and must obviously be equal to t because of ξ being $\dfrac{h}{2}$.

Next consider the case of h lying in $(\frac{1}{2}, 1)$. *Assume* that

$$w(t)=a_0+a_1t+a_2t^2+\ldots, \tfrac{1}{2}\leq t\leq 1.$$

Then, because of the continuity of $w(t)$ at $t=\frac{1}{2}$,

$$\tfrac{1}{2}=a_0+a_1.\tfrac{1}{2}+a_2.\frac{1}{2^2}+\ldots \tag{I}$$

Also,

$$f(h)=\int_0^{\frac{1}{2}} w(t)dt+\int_{\frac{1}{2}}^{h} w(t)\,dt=\tfrac{1}{2}.(\tfrac{1}{2})^2+\Big[a_0t+\frac{a_1}{2}t^2+\ldots\Big]_{\frac{1}{2}}^{h}$$

$$=hw(\xi)=h\xi \text{ or } h\{a_0\xi+a_1\xi+\ldots\}, \tag{II}$$

according as ξ is in $(0, \frac{1}{2})$ or in $(\frac{1}{2}, 1)$

From the first alternative of (II), we have

$$\Big(a_0h+\frac{a_1}{2}h^2+\ldots\Big)-a_0.\tfrac{1}{2}+\frac{(1-a_1)}{2}.\frac{1}{2^2}-\ldots=h.\frac{h-\frac{1}{2}h^2-\frac{1}{4}}{h}$$

Hence $a_0=1$, $a_1=-1$, $a_2=a_3=\ldots=0$.

Then values of the a's satisfy the second alternative of (II) as well as (I). Thus $w(t)$ is t or $1-t$ according as t is in $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$. And

$f(h)=\frac{h^2}{2}$ or $h-\frac{h^2}{2}-\frac{1}{4}$ according as h is in $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$.

Ex. 2. If ξ is given by the following scheme, find $f(h)$ on the same assumption as in *Ex.* 1 :—

Dividing the interval (0, 1) into four quarters $\left(0, \frac{1}{2^2}\right), \left(\frac{1}{2^2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{3}{2^2}\right), \left(\frac{3}{2^2}, 1\right)$ and calling them respectively the first, second, third and fourth quarters; ξ is equal to $\frac{h}{2}$ when h is in the first quarter; for values of h in the second quarter ξ or $\frac{1}{2}-\xi$ equals

$$\frac{-\frac{h^2}{2}+\frac{1}{2}h-\frac{1}{2^4}}{h}$$

according as ξ is in the first or second quarter; for values of h in the third quarter ξ, $\frac{1}{2}-\xi$ or $\xi-\frac{1}{2}$ equals

$$\frac{\frac{h^2}{2}-\frac{1}{2}h+\frac{3}{2^4}}{h}$$

according as ξ is in the first, second or third quarter; and lastly for values of h in the fourth quarter ξ, $\frac{1}{2}-\xi$, $\xi-\frac{1}{2}$ or $1-\xi$ equals

$$\frac{-\frac{h^2}{2}+h-\frac{3}{8}}{h}$$

according as ξ is in the first, second, third or fourth quarter.

(*b*) For the case in which $\theta(h)$ has *infinite* number of values, see Prasad's paper "On the determination of $f(h)$ corresponding to a given Rolle's function $\theta(h)$ when it is multiple-valued" (*Proc. Benares M. S.*, Vol. XII).

SIXTH LECTURE

ROLLE'S FUNCTION θ AS A FUNCTION OF x AND h ; GENERALIZED MEAN-VALUE THEOREMS, GENERALIZED FUNCTIONS θ.

§ 41

65. To-day I will first discuss the Rolle's function θ as a function of *two variables, viz.*, x and h, and then proceed to give a number of miscellaneous theorems and results, each of which is, in some sense, connected with the mean-value theorem or the Rolle's function θ as a kind of generalization. Taking up θ (x, h), I proceed to formulate and prove in the words of Rothe the following three theorems:

Theorem I: Let x and h $(\neq 0)$ be real, $a \leq x \leq b$ and also $a \leq x+h \leq b$; further let $f(x)$ be continuous for $a \leq x \leq b$, and differentiable for $a < x < b$. Then, of all the functions $f(x)$ satisfying the above conditions, $f(x) = \alpha x^2 + \beta x + \gamma$ $(\alpha \neq 0)$ alone has the property to satisfy the mean-value theorem

$$f(x+h) - f(x) = hf'(x+\theta h), \qquad (\mathrm{M}_1)$$

with a value of θ *independent of x and h, viz.*, $\theta = \frac{1}{2}$.

Proof:

"If x lies in the inside of (a, b), then is the left side and therefore also the right side of (M_1) differentiable with respect to x ; consequently is $f(x)$, because of the constancy of θ twice differentiable. By differentiating both sides of (M_1) with respect to x and with respect to h, we have

$$(1) \quad \begin{cases} f'(x+h) - f'(x) = h\, f''(x+\theta h), \\ f'(x+h) \qquad\qquad = h\, f''(x+\theta h).\, \theta + f'(x+\theta h). \end{cases}$$

From these two equations follows immediately

$$f'(x+h)\,(1-\theta) + \theta f'(x) = f'(x+\theta h).$$

Differentiating the above once *more with respect to h*, we have

$$(2) \qquad f''(x+h)\,(1-\theta) \qquad = f''(x+\theta h).\, \theta$$

Now $f'(x)$ is differentiable in the inside of (a, b) and therefore continuous, and consequently according to (1) $f''(x)$ is also continuous there. Consequently from (2) we have for $h \to 0$

$$f''(x)\,(1-2\theta) = 0.$$

If now $f''(x)$ were identically equal to 0, then $f(x)$ would be linear, and the formula (M_1) would be satisfied for every arbitrary value of θ. This trivial case shall be here, as also in the following pages, excluded.

Therefore θ must be $\frac{1}{2}$.

Thus from (2) we have

$$f''(x+h)=f''\left(x+\frac{h}{2}\right).$$

Since h is permitted to take arbitrary values, for which $x+\frac{h}{2}$ and $x+h$ are in the inside of (a, b), therefore it follows that $f''(x)$ must be a constant different from 0, and therefore $f(x)=\alpha x^2+\beta x+\gamma$, α, β, γ being constants.

66. *Theorem II*: If in the mean-value theorem (M_1), $\theta=\theta(h)$ be *independent of x and dependent on h alone*, and further $\theta(h)$ *be single-valued and differentiable*; then, of all the functions $f(x)$ which are continuous in (a, b), the ends being included, and differentiable in (a, b), the ends being excluded, $f(x)=ce^{\alpha x}\ \beta x+\gamma$ alone has the property to satisfy (M_1) with such a θ, as stated above, which is *not constant*. This $\theta(h)$ is $\frac{1}{\alpha h}\log\frac{e^{\alpha h}-1}{\alpha h}$.

Proof:

First, it follows from (M_1), by the same considerations as before, that because θ does not depend on x, $f''(x)$ and therefore also $f'''(x)$, ...exist for $a<x<b$. By differentiation with respect to x and with respect to h, we have from (M_1), putting ξ for θh,

$$(3)\qquad \begin{cases} f'(x+h)-f'(x)=h\,f''(x+\xi), \\ f'(x+h) \qquad\quad =f'(x+\xi)+hf''(x+\xi)\xi', \end{cases}$$

where $\xi'=\frac{d\xi}{dh}$. One can assume that ξ' is not identically zero, for otherwise would ξ be constant, and from the above equation would also $f'(x)$ be constant which would lead to the trivial case of $f(x)$ as a linear function. Therefore it follows from the two equations by the elimination of $f''(x+\xi)$, as is easily seen, that

$$(4)\qquad f'(x+h)(1-\xi')+\xi'f'(x)=f'(x+\xi)$$

or

$$\xi'\{f'(x+h)-f'(x)\}=f'(x+h)-f'(x+\xi).$$

Now however this equation can be always solved [1] with respect to

[1] If $f'(x+h)-f'(x)=0$ for arbitrary values of h and of x, then must $f'(x)$ be constant in the whole region, a possibility which was excluded at the outset by excluding linear $f(x)$. However, if, for a definite value of h and arbitrary values of x, $f'(x+h)=f'(x)$, *i.e.*, if $f'(x)$

ξ'.....;Consequently (4) admits of a solution with respect to ξ', *viz.*,

$$\xi' = \frac{f'(x+h) - f'(x+\xi)}{f'(x+h) - f'(x)}$$

The right side, which as already said is independent of x, is differentiable with respect to h. Therefore ξ'' exists.

Moreover, follows from (4) by differentiation with respect to h:

$$f''(x+h)\,(1-\xi') - \xi''\{f'(x+h) - f'(x)\} = f''(x+\xi)\,.\,\xi'$$

or, by using (3),

(5_1) $f''(x+h)\,(1-\xi') - f''(x+\xi)(h\xi'' + \xi') = 0,$

and therefore by further differentiation with respect to x:

(5_2) $f'''(x+h)(1-\xi') - f'''\,(x+\xi)(h\xi'' + \xi') = 0.$

Now, however, as is immediately clear, the quantities $1-\xi'$ and $h\xi'' + \xi'$ cannot be both identically 0. Therefore the determinant of the two equations (5_1) and (5_2)

$$\begin{vmatrix} f''(x+h) & f''(x+\xi) \\ f'''(x+h) & f'''(x+\xi) \end{vmatrix} = 0.$$

But $f''(x+\xi)$ cannot be zero, for then according to (3) must $f'(x+h) - f'(x)$ vanish which, because of the previous considerations does not happen. Hence $f''(x+h)$ also cannot be zero as a glance at the equation (5_1) shows. Consequently the above determinantal equation admits of being written in the form

$$\frac{f'''\,(x+h)}{f''\,(x+h)} = \frac{f'''\,(x+\xi)}{f''\,(x+\xi)}.$$

Since $\xi = \theta h$ cannot be equal to h because of $\theta < 1$, therefore one may put

$$\xi = h + p(h)$$

where $p(h) \neq 0$. Therefore the preceding equation, with $x+h=z$, changes into

$$(6) \qquad \frac{f'''\,(z)}{f''\,(z)} = \frac{f'''\,(z+p)}{f''(z+p)}.$$

But p cannot be an absolute constant, for otherwise $\xi' = 1$, $\xi'' = 0$ and therefore by (5_1) $f''(x+\xi) = f''(z+p) = 0$ which would again lead to the case

had the period h, then it would follow from (4) that also $f'(x) = f'(x+\xi) = f'(x+\theta h)$ *i.e.* $f'(x)$ would also have the period θh which is less than h. By repeatedly proceeding as above, it will follow that $f'(x)$ has an arbitrarily small period, and must therefore be constant. So it follows that $f'(x+h) - f'(x) \neq 0$ and so (4) admits of a solution.

of linear $f(x)$. Therefore $p=p(h)$ is variable, and, since ξ is differentiable and therefore continuous, the same holds for $p(h)$, and therefore $p(h)$ can take all the values of a certain domain. This leads, however, with respect to the equation (6), to the result that $\frac{f'''(z)}{f''(z)}$ must be constant. If, one writes x in the place of z,

$$\frac{f'''(x)}{f''(x)}=C,$$

where C denotes a constant. From this equation, $f(x)$ is easily determined."

$$f(x)=C_3\,e^{Cx}-\frac{C_1}{C}\left(x+\frac{1}{C}\right)-\frac{C_2}{C}\quad\text{for } C\neq 0,$$

for $C=0$:

$$f(x)=C_1\,\frac{x^2}{2}+C_2x+C_3$$

C_1, C_2, C_3 being constants.

Leaving aside the case of $C=0$ which gives θ independent of h, we have the expression $f(x)=ce^{ax}+\beta x+\gamma$ with $a\neq 0$.

67. *Theorem III*: Of all the functions $f(x)$ which are continuous in (a, b), the ends being included and differentiable in (a, b) the ends being excluded, there is no function for which the theorem (M_1) is satisfied with a *θ independent of h, dependent only on x and possessing* $\frac{d\theta}{dx}$.

Proof: "The left side of the equation

$$\frac{f(x+h)-f(x)}{h}=f'(\xi),\ (\xi=x+\theta h)$$

is, for $h\neq 0$ and for every fixed value of x in the interval $a<x<b$, continuous and differentiable with respect to h, and since the argument ξ as a linear function of h is likewise continuous and differentiable with respect to h for every fixed value of x, therefore the same holds for $f'(\xi)$, *i.e.*, $f''(x)$ exists in the interval $a<x<b$. Moreover, the differentiation of (M_1) with respect to h and with respect to x gives the two equations

$$f'(x+h)=f'(x+\theta h)+hf''(x+\theta h).\ \theta,$$
$$f'(x+h)-f'(x)=h\,f''(x+\theta h)(1+\theta' h).$$

From the first follows at once, because of $0<\theta$, that for $h\neq 0$ also $f''(\xi)$ is continuous, *i.e.*, $f''(x)$ is continuous for $a<x<b$.

In the second equation, the factor of $f''(x+\theta h)$ cannot vanish for every value of h. Consequently, $f''(x+\theta h)$ admits of being eliminated and one obtains thus

$$(1+\theta' h-\theta)\, f'(x+h)+\theta f'(x)=(1+\theta' h)\, f'(x+\theta h).$$

One more differentiation with respect to h gives

$$\theta' f'(x+h)+(1+\theta' h-\theta)\, f''(x+h)=\theta' f'(x+\theta h)+(1+\theta' h)\, f''(x+\theta h).\,\theta.$$

Now it is expedient to let h converge to 0. Thereby one obtains, by having regard to the continuity of $f''(x)$,

$$(1-\theta)\, f''(x)=f''(x).\theta$$

and, since the identical vanishing of $f''(x)$ would lead to linear $f(x)$, there remains the only possibility

$$\theta=\tfrac{1}{2},$$

i.e., θ is the same constant as in Theorem I."

§ 42

68. I proceed now to give briefly Rothe's treatment of the question: What conditions must be satisfied by θ as a function of x *and* h in order that there should be a corresponding function $f(x)$ to satisfy the mean-value theorem

$$f(x+h)=f(x)+h\, f'(x+\theta h)\,?$$

"Let θ be a function, subject to the condition $0<\theta<1$, of the two independent variables $x_1=x$ and $x_2=x+h$ which take values inside the quadratic region

$$a<x_1<b,\ a\lessgtr x_2<b.$$

The function $\xi=\xi(x_1, x_2)=x_1+\theta(x_1, x_2)(x_2-x_1)$ takes only values of the interval $a<\xi<b$. The question is: What conditions must be satisfied by ξ as a function of x_1, x_2 in order that there be a function $f(x)$ which is continuous for $a\leq x\leq b$, differentiable for $a<x<b$ and satisfies the equation

(M) $$f(x_2)-f(x_1)=(x_2-x_1)\, f'(\xi)?"$$

Let those pairs x_1, x_2 be excluded for which the following conditions I–IV are not satisfied and let the remaining part of the region be denoted by B:—

I. $x_1 \neq x_2$, *i.e.*, $h=x_2-x_1\neq 0$.

II. $\theta(x_1, x_2)$ possesses the continuous partial differential co-

efficients $\dfrac{\partial\theta}{\partial x_1}=\theta_1$, $\dfrac{\partial\theta}{\partial x_2}=\theta_2$, $\dfrac{\partial\theta_1}{\partial x_2}=\dfrac{\partial\theta_2}{\partial x_1}=\theta_{12}$, $\dfrac{\partial\theta_{12}}{\partial x_1}=\theta_{121}$, $\dfrac{\partial\theta_{12}}{\partial x_2}$

$$=\theta_{122}.$$

It is easily seen that the condition II is also satisfied by $\xi(x_1, x_2)$. The partial differential co-efficients $\dfrac{\partial\xi}{\partial x_1}=\xi_1$, $\dfrac{\partial\xi}{\partial x_2}=\xi_2$ cannot be both identically 0 ; for, otherwise, ξ would be a constant and therefore also $f'(\xi)$ which would give according to (M) a linear $f(x)$ which case was excluded beforehand.

Now the left side of (M), and, consequently also, the right side are partially differentiable with respect to x_1, as well as x_2, and, because by II, ξ_1 and ξ_2 are existent, therefore $f''(\xi)$ also exists and equals $\dfrac{\partial f'(\xi)}{\partial x_1}\Big/\xi_1=\dfrac{\partial f'(\xi)}{\partial x_2}\Big/\xi_2$, it being assumed that ξ_1 and ξ_2 are both different from zero. Under this supposition, by the partial differentiation of (M) we have

$$(\mathrm{M}')\left\{\begin{aligned} f'(x_2)&=(x_2-x_1)\,f''(\xi).\,\xi_2+f'(\xi),\\ -f'(x_1)&=(x_2-x_1)\,f''(\xi).\,\xi_1-f'(\xi),\end{aligned}\right.$$

and hence because of I

$$f''(\xi)=\frac{f'(x_2)-f'(\xi)}{(x_2-x_1).\xi_2}=-\frac{f'(x_1)-f'(\xi)}{(x_2-x_1).\xi_1}$$

III. Neither ξ_2 nor ξ_1 vanish.

The expression for $f''(\xi)$ admits of partial differential co-efficients in the whole of the region (B) and because of

$$f'''(\xi)=\frac{\partial f''(\xi)}{\partial x_1}\Big/\xi_1=\frac{\partial f''(\xi)}{\partial x_2}\Big/\xi_2,$$

$f'''(\xi)$ is existent in the whole of (B). Now this being settled, by differentiating with respect to x_1 both the sides of the first equation, (M′) or with respect to x_2 both the sides of the second equation, we have

$$(\mathrm{M}'')\quad (x_2-x_1)f''(\xi).\quad \xi_{12}+(x_2-x_1)f'''(\xi).\quad \xi_1\xi_2+f''\xi(\xi_1-\xi_2)=0$$

where

$$\xi_{12}=\frac{\partial\xi_1}{\partial x_2}=\frac{\partial\xi_2}{\partial x_1}.$$

IV. $f''(\xi) \neq 0$.

This supposition enables one to write (M″) in the form

$$\frac{f'''(\xi)}{f''(\xi)} = -\zeta \;(\alpha), \text{ where } \zeta = \frac{(x_2 - x_1)\xi_{12} + (\xi_1 - \xi_2)}{(x_2 - x_1)\xi_1 \xi_2} \quad (\beta).$$

The function ζ exists in the whole of (B) and admits clearly of the partial differential co-efficients $\dfrac{\partial \zeta}{\partial x_1} = \zeta_1$, $\dfrac{\partial \zeta}{\partial x_2} = \zeta_2$. Hence it follows from (α) that in (B) $f^{(4)}(\xi)$ exists and one obtains by partial differentiation with respect to x_1 and x_2 and the elimination of $f^{(4)}(\xi)$

$$f''(\xi)(\xi_1\zeta_2 - \xi_2\zeta_1) = 0$$

whence by IV we have $\xi_1\zeta_2 - \xi_2\zeta_1 = 0$, a partial differential equation * of the 3rd order as the necessary condition for the existence of $f(x)$.

69. The satisfaction of the above differential equation is, however, not sufficient for the existence of $f(x)$; as the following example proves:

Let θ be given by

$$e^{a \cdot h\theta} = \frac{A\, e^{ah} + B}{ah}, \qquad \ldots \quad (1)$$

where a, A, B are constants. Then it is easily seen that

$$\zeta = -a \qquad \ldots \quad (2)$$

whatever A, B *may be*. So the differential equation (2) is satisfied whatever A, B may be; but from what was proved in Theorem II it is clear that θ being a function of h alone, in order that there should be a corresponding $f(x)$, we must have

$$A = 1, \; B = -1.$$

For any other values, the function θ gives an $\xi = x_1 + h\theta$, where $h = x_2 - x_1$, which satisfies (2) but does not correspond to any $f(x)$.

* Putting $\dfrac{\partial \xi}{\partial x_1} = p$, $\dfrac{\partial \xi}{\partial x_2} = q$, ... $\dfrac{\partial^2 \xi}{\partial x_2^2} = t$, we have the equation in the form

$$pq\left(p\frac{\partial t}{\partial x_1} - q\frac{\partial r}{\partial x_2}\right) - s\,(p^2 t - q^2 r) - \frac{q^2 r - pq(p+q)s + p^2 t}{x_2 - x_1}$$

$$- pq\frac{p^2 - q^2}{(x_2 - x_1)^2} = 0.$$

§ 43.

70. The following theorems, the proof of each of which is briefly indicated, may be considered to be generalizations of the mean-value theorem in the sense that each of them depends for its validity on Rolle's theorem and gives the mean-value theorem as a particular case:

(a) *Lagrange's remainder theorem:*

$$f(x+h)=f(x)+hf'(x)+\ldots+\frac{h^{n-1}f^{(n-1)}(x)}{(n-1)!}+\frac{h^n f^{(n)}(x+\theta h)}{n!},\ 0<\theta<1,\ n\geq 1.$$

Let $\psi(t)$ denote $f(b)-f(t)-(b-t)f'(t)-\frac{(b-t)^2}{2!}f''(t)-\ldots$

$$-\frac{(b-t)^{n-1}}{(n-1)!}f^{(n-1)}(t)-\frac{(b-t)^n}{n!}\mathrm{P},$$

where $b=x+h$, and P is independent of t and is given by

$$f(x+h)=f(x)+hf'(x)+\ldots+\frac{h^{n-1}f^{(n-1)}(x)}{(n-1)!}+\frac{h^n}{n!}\mathrm{P}.$$

Then $\psi(x)=0$, $\psi(x+h)=0$, and if $f^{(n)}(t)$ exists for every value of t inside $(x, x+h)$, Rolle's theorem gives

$$\psi'(t)=0$$

for a value of t between x and $(x+h)$. But

$$\psi'(t)=\frac{(b-t)^{n-1}}{(n-1)!}\{\mathrm{P}-f^{(n)}(t)\}.$$

Therefore we have

$$\psi'(x+\theta h)=0,\ i.e.,\ \mathrm{P}=f^{(n)}(x+\theta h).$$

(b) *Cauchy's generalized mean-value theorem:*

$$\frac{\phi(x+h)-\phi(x)}{(\mathrm{F}x+h)-\mathrm{F}(x)}=\frac{\phi'(x+\theta h)}{\mathrm{F}'(x+\theta h)},\ 0<\theta<1.$$

Let

$$\psi(t)=\phi(t)-\phi(x)-\frac{\phi(x+h)-\phi(x)}{\mathrm{F}(x+h)-\mathrm{F}(x)}\{\mathrm{F}(t)-\mathrm{F}(x)\},$$

then $\psi(x)=0$, $\psi(x+h)=0$; therefore Rolle's theorem gives

$$\psi'(t)=0$$

for a value of t between x and $x+h$, if $\phi'(t)$ and $\mathrm{F}'(t)$ exist for every value of t inside $(x, x+h)$, and $\mathrm{F}'(t)$ is nowhere 0 or infinite. Hence the theorem.

(c) *Genocchi and Peano's generalized mean-value theorem:*

$$\begin{vmatrix} f(x+h) & \phi(x+h) & \mathrm{F}(x+h) \\ f(x) & \phi(x) & \mathrm{F}(x) \\ f'(x+\theta h) & \phi'(x+\theta h) & \mathrm{F}'(x+\theta h) \end{vmatrix}=0,\ 0<\theta<1.$$

Let

$$\psi(t)=\begin{vmatrix} f(x+h) & \phi(x+h) & F(x+h) \\ f(x) & \phi(x) & F(x) \\ f(t) & \phi(t) & F(t) \end{vmatrix},$$

then $\psi(x)=0$, $\psi(x+h)=0$; therefore Rolle's theorem gives

$$\psi'(t)=0$$

for a value of t between x and $x+h$, if $f'(t)$, $\phi'(t)$, $F'(t)$ exist for every value of t in $(x, x+h)$ and do not all vanish simultaneously. Hence the theorem. It may be noted that for $F(t)=1$ we have Cauchy's generalized theorem and from this for $\phi(t)=t$ the ordinary mean-value theorem.

(c′) Generally[1] for $(n+1)$ functions $f_0(x)$, $f_1(x)$, $f_2(x)$,....$f_n(x)$, we have

$$\begin{vmatrix} f_0^{(n-1)}(u) & f_1^{(n-1)}(u) & \dots & \dots & f_n^{(n-1)}(u) \\ f_0(x_1) & f_1(x_1) & \dots & \dots & f_n(x_1) \\ f_0(x_2) & f_1(x_1) & \dots & \dots & f_n(x_2) \\ \dots & \dots & \dots & \dots & \dots \\ f_0(x_n) & f_1(x_n) & \dots & \dots & f_n(x_n) \end{vmatrix}=0,$$

where u is a mean of $x_1, x_2, \dots, x_n$.

(d) *Pompeiu's theorem.*[2]

$$\{\phi(x+h)-\phi(x)\}\{F(x+h)-F(x)\}=h^2\phi'(x+\theta h)F'(x+\theta h),\quad 0<\theta<1.$$

The theorem is proved by Pompeiu on the supposition that ϕ' and F' are positive and increasing.

(e) $f(x+h,\ y+k)=f(x,\ y)+h\phi(x+\theta h,\ y+\theta k)+k\psi(x+\theta h,\ y+\theta k),$
$0<\theta<1;$

$\phi(u, v)$ standing for

$$\frac{\partial}{\partial u}f(u,\ v),\ \psi(u,\ v)\ \text{ for } \frac{\partial}{\partial v}f(u,\ v).$$

Consider the function $\chi(t)$ of t, *viz.*,

$$\chi(t)=f(x+ht,\ y+kt),\quad 0\leq t\leq 1.$$

Then, by the ordinary mean-value theorem,

$$\chi(1)=\chi(0)+\chi'(\theta),\quad 0<\theta<1,$$

and

$$\chi',\ i.e.,\ \frac{d\chi}{dt}=\frac{\partial f}{\partial u}\cdot h+\frac{\partial f}{\partial v}\cdot k$$

[1] Genocchi : Peano's *Calcolo Differenziale*, p. xxii.

[2] "Sur une forme du théorème des accroissements finis," *Bulletin de l' Academie Roumaine*, 1924.

Thus

$$\chi'(\theta)=h\phi(x+\theta h,\ y+\theta k)+k\psi(x+\theta h,\ y+\theta k).$$

(*f*) *Goursat and Hedrick's form*,[1] *viz.*,

$$f(x+h,\ y+k)=f(x,\ y)+hf'_x(x+\theta h,\ y+k)+kf'_y(x,\ y+\theta k)$$

follows by considering

$$\chi(t)=f(x+ht,\ y+k)+f(x,\ y+kt).$$

§ 44.

71. The number θ which occurs in any of the generalized theorems of § 43 can be studied in the same manner as the θ which occurs in the ordinary mean-value theorem; I will conclude this lecture with a brief account of the researches relating to the other θ's:

(*a*)(*i*) The number θ in Lagrange's remainder was probably studied by Cauchy and his school with the result that $\theta(+0)$ was shown to be $\dfrac{1}{n+1}$; but the first published investigation about θ is due to Whitcom [2] who gave the first four terms in the expansion of $\theta(h)$, these being given as follows;

$$\theta=\frac{1}{n+1}+h\left[\left\{\frac{2(n+1)-(n+2)}{2(n+1)^2(n+2)}\right\}\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\right]$$

$$+\frac{h^2}{2!}\left[\left\{\frac{2\cdot3\cdot(n+1)^2-(n+2)(n+3)}{3(n+1)^3(n+2)(n+3)}\right\}\frac{f^{(n+3)}(x)}{f^{(n+1)}(x)}\right.$$

$$\left.-\left\{\frac{2(n+1)-(n+2)}{(n+1)^3(n+2)}\right\}\left\{\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\right\}^2\right]$$

$$+\frac{h^3}{3!}\left[\left\{\frac{2\cdot3\cdot4(n+1)^3-(n+2)(n+3)(n+4)}{2^2(n+1)^4(n+2)(n+3)(n+4)}\right\}\frac{f^{(n+4)}(x)}{f^{(n+1)}(x)}\right.$$

$$-\left\{\frac{2\cdot3\cdot(n+1)-3(n+2)}{2(n+1)^4(n+2)}+\frac{2\cdot3^2(n+1)^2-3(n+2)(n+3)}{3(n+1)^4(n+2)(n+3)}\right\}$$

$$\left\{\frac{f^{(n+3)}(x)\ .\ f^{(n+2)}(x)}{f^{(n+1)}(x)\ .\ f^{(n+1)}(x)}\right\}$$

$$-\left\{\frac{2^2\cdot3(n+1)^2-2^2\cdot3(n+1)(n+2)+3(n+2)^2}{2^2\cdot(n+1)^4(n+2)^2}\right.$$

$$\left.\left.-\frac{2\cdot3(n+1)-3(n+2)}{(n+1)^4(n+2)}\right\}\left\{\frac{f^{(n+2)}(x)}{f^{(n+1)}(x)}\right\}^2\right]-\cdots\cdots$$

[1] *Mathematical Analysis*, Vol. 1.

[2] *L.c.*, p. 351.

(*ii*) As stated in the first lecture, Rothe and Hayashi have studied this θ and Rothe has given a number of results similar to those for Rolle's function θ.

(*iii*) Dini [1] has shown that, if, at $t=x$, $f^{(n+1)}(t)$ exists, is finite and $\neq 0$, $\theta(+0)$ exists and equals $\frac{1}{n+1}$.

(*b*) (*i*) The number θ occurring in Cauchy's generalized theorem was apparently studied at Cambridge as early as 1894 for which year there is a question paper [2] containing the question:

"If $F(x)$ denote $\dfrac{f'(x)\phi'''(x)-\phi'(x)f'''(x)}{f'(x)\phi''(x)-\phi'(x)f''(x)}$, prove that

$$\theta=\frac{1}{2}+\frac{h}{24}F(x) \text{ approximately when } h \text{ is small."}$$

In the above θ is given by

$$\frac{f(x+h)-f(x)}{\phi(x+h)-\phi(x)}=\frac{f'(x+\theta h)}{\phi'(x+\theta h)}.$$

(*ii*) As Rothe considered the case in which Rolle's function $\theta(x, h)$ is an absolute constant, so Takahashi has considered the case in which the θ in Cauchy's generalized theorem is an absolute constant.[3] He has proved that θ must be $\frac{1}{2}$ and that, f and ϕ being assumed to be differentiable five times, they must belong to only the following three types:

(1) $f=Ax^2+Bx+c,\ \phi=A'x^2+B'x+c'$

(2) $f=Ae^{px}+Be^{-px}+c,\ \phi=A'e^{px}+B'^{-px}+c'$

(3) $f=A\sin(px+q)+B,\qquad \phi=A'\sin(px+r)+B'$

or $\qquad\qquad$ or

$A\cos(px+q)+B,\qquad A'\cos(px+r)+B'$

In the above $c=0$ in (1), $c>0$ in (2) and $c<0$ in (3); all the other constants are arbitrary but different from 0.

(*c*) Takahashi has considered a question similar to the above for the θ in Genocchi and Peano's theorem.[4]

[1] "Calcolo Infinitesimale," t. 1, p. 69, foot-note.

[2] Of Trinity College, June 6, 1894.

[3] *L.c.*, pp. 436-439.

[4] *L.c.*, pp. 439-440.

APPENDIX A.

On Pompeiu's Proof of the Mean-value Theorem.

Universite' de Bucarest,
le 15 Mars, 1931.

Très honoré Monsieur et Professeur,

Je vieus de recevoir votre lettre ainsi que l'extrait de votre livre où (§ 15) vous me faites l'honneur de donner une place à ma méthode pour démontrer le théoréme des accroissements finis.

Veuillez en accepter tous mes remerciements très sincères.

Vous faites suivre l'expose de ma démonstration de quelques observations (criticism of Pompeiu's proof) sur lesquelles vous désirez avoir aussi mon opinion.

Je suis très sensible à la délicatesse de ce procédé.

Voici quelle est mon opinion :

Le *fait*, que vous signalez par votre exemple (page 28) est *exact* et, de plus, c'est un *fait* général dans le cas d'une fonction dérivee quelconque c'est-à-dire pourvue de points de discontinuite'.

Mais, ce fait n' intervient pas dans ma démonstration.

Dans ma démonstration le point c est toujours *intérieur* aux intervalles (x_k, y_k) et alors *le fait signalé*, par votre example, *ne peut pas se produire*.

Dans votre exemple le point $c=\frac{1}{2}$ est *extérieur* aux intervalles

$$x_k=\tfrac{1}{2}+\frac{1}{2^k\pi}, \quad y_k=\tfrac{1}{2}+\frac{1}{2^k\pi-\frac{\pi}{2}};$$

c'est pour cela que le fait, signalé par vous est possible.

En résumé : vous signale, par votre exemple, un fait *exact* ; mais ce fait n'a rien à voir dans ma démonstration, parce que les circonstances sont autres dans ma méthode : le point c est toujours *intérieur* aux intervalles (x_k, y_k) es alons le fait, en question, ne peut pas se produire.

Et, puisque, ainsi éclaircie, la question présente un intéret scientifique, j'ai rédigé une Note (ci-jointe) et, si vous trouves cela convenable, vous pouvez (apres traduction en anglais, ou meme sous sa forme primitive) la faire figurer, à la fin de votre livre, comme une simple Note explicative.

Je vous en serais très obligé, pour l'intérêt scientifique de la question.

Veuillez agréer, tres honoré Monsieur le Professeur, l'assurance de ma considération tres distinguée et me croire

votre tout dévoué

D. POMPEIU,

Agréé à l'Université de Paris.

NOTE.

Sur une propriété des fonctions dérivées.

par D. POMPEIU.

1. Soit $f(x)$ une fonction, définie dans un intervalle (a, b) et admettant pour tout point x, intérieur à (a, b) une dérivée bien déterminée $f'(x)$.

Cela veut dire que: si x est un point fixe, pris dans (a, b) le rapport

$$\frac{f(x+h)-f(x)}{h}$$

a une limite bien déterminée, lorsque h tend vers zéro (par valeurs positives ou négatives).

2. Cela précisé, soit c un point pris dans l'intérieur de (a, b) et (x_k, y_k) une suite d'intervalles, d'amplitude de plus en plus petite, et tendant vers le point c.

Formons les rapports

$$\frac{f(x_k)-f(y_k)}{x_k-y_k}=\mathrm{R}(x_k, y_k).$$

Que peut-on dire de la *limite* des rapports $\mathrm{R}(x_k, y_k)$ lorsque les intervalles (x_k, y_k) tendent vers le point c?

Voici la réponse précise:

1° Cette limite peut ne pas exister du tout. On peut, facilement, former un exemple avec la fonction dérivée

$$f'(x) = \sin\frac{1}{x}$$

dans le voisinage du point $x=0$.

2° Cette limite peut exister et être différente de la valeur de $f'(x)$ au point $x=c$.

C'est l'exemple signalé à la page 28 de ce livre.

3° Cette limite existe et est égale à la dérivée $f'(c)$ si tous les intervalles (x_k, y_k) contiennent le point c à leur *intérieur*.

Cela résulte immédiatement de l'identité

$$\frac{f(x_k)-f(y_k)}{x_k-y_k}=\frac{f(x_k)-f(c)}{x_k-c}\cdot\frac{x_k-c}{x_k-y_k}+\frac{f(c)-f(y_k)}{c-y_k}\cdot\frac{c-y_k}{x_k-y_k}.$$

Puisque c est intérieur à (x_k, y_k), on a

$$\left|\frac{x_k-c}{x_k-y_k}\right|<1 \quad \left|\frac{c-y_k}{x_k-y_k}\right|<1$$

et alors le rapport du premier membre est *compris entre* les deux rapports qui figurent au second membre.

C'est ce fait seul qui se présente dans la démonstration, du théorème des accroissements finis, exposée aux pages 26, 27 de ce livre.

3. Mais, il résulte de ce qui précède que la dérivée d'une fonction $f(x)$ peut être définie :

soit par la limite du rapport classique

$$\text{(1)} \qquad \frac{f(x+h)-f(x)}{h}.$$

soit par le rapport

$$\text{(2)} \qquad \frac{f(x')-f(x'')}{x'-x''}$$

avec la condition expresse que le point x où l'on veut avoir la dérivée, soit constamment compris entre x' et x'' :

$$x'<x<x''.$$

Alors la dérivée cherchée est aussi bien la limite du rapport (1) que la limite du rapport (2)

D. Pompeiu.

Remarks on the above

By

Ganesh Prasad

(I)

The fact that $\frac{f(x_k)-f(y_k)}{x_k-y_k}$ tends to a limit as x_k and y_k both tend to c, even if, for every value of k,

$$x_k<c<y_k,$$

does not carry us far; as will be clear from the following examples, in each of which $a=-1$, $b=1$.

Example 1. Let $f(x)=x^{\frac{2}{3}}$ and take $x_k=-y_k$, x_k tending to 0 in any manner whatsoever.

Then $f(x_k)-f(y_k)$ is zero for every value of k; consequently, x_k and y_k both tend to 0, while $\dfrac{f(x_k)-f(y_k)}{x_k-y_k}$ also tends to 0.

But there is no differential co-efficient for $f(x)$ at $x=0$; and, as a matter of fact, the mean-value theorem does *not* hold.

Example 2. Let $f'(x)=\sin\dfrac{1}{x}$, so that

$$f(x)=\int_0^x \sin\frac{1}{t}\,dt;$$

and take $x_k=-y_k$, x_k tending to 0 in any manner whatsoever.

Contrary to Professor Pompeiu's statement under 1° in his *Note*, $f'(0)$ exists [1] and is 0.

Also $f(x_k)=f(y_k)$, as $f(x)$ is an even function; consequently, x_k and y_k both tend to 0, while $\dfrac{f(x_k)-f(y_k)}{x_k-y_k}$ also tends to 0.

Also, there is a differential co-efficient for $f(x)$ at $x=0$; and, as a matter of fact, the mean-value theorem holds.

Example 3. Let $f(x)=x\sin\dfrac{1}{x}$ and take $x_k=-y_k$, x_k tending to 0 in any manner whatsoever.

Then, $f(x)$ being an even function, $f(x_k)-f(y_k)$ is zero for every value of k; consequently, x_k and y_k both tend to 0, while

$$\frac{f(x_k)-f(y_k)}{x_k-y_k}$$

also tends to 0.

Although there is no differential co-efficient at $x=0$, the mean-value theorem holds.[2]

[1] This is easily seen as follows:

$$f(x)=\int_0^x t^2\frac{d}{dt}\left(\cos\frac{1}{t}\right)dt=x^2\cos\frac{1}{x}-\int_0^x 2t\cos\frac{1}{t}dt.$$

Therefore $f'(0)$, which equals the sum of the differential co-efficients at $x=0$, of

$$x^2\cos\frac{1}{x}\text{ and}-\int_0^x 2t\cos\frac{1}{t}dt,$$

is zero, as each of the two has a differential co-efficient 0 at $x=0$.

[2] See Art. 28.

(II)

Leaving now the Note of Professor Pompeiu given in this Appendix, and going back to the proof reproduced in § 15, I wish to emphasize that, like the classical proof of Bonnet or Dini, it is based on the following property [1] of continuous functions:

"If the function $F(x)$ is continuous between a and b and in two points α and β of this interval (a and b being included) takes different values A and B, then for one or more determinate values of x between α and β, $F(x)$ shall take any value C comprised between A and B."

For the purposes of Bonnet's or Dini's proof, the above property will do with the ordinary notion of continuity in which $F(x)$ is *bounded*. But with such a notion, Professor Pompeiu's proof will fail unless the $f(x)$ of § 15 is restricted to possess only *finite* differential co-efficients.

For, for the purposes of Bonnet's or Dini's proof, $F(x)$ is practically $f(x)$, whereas for Professor Pompeiu's proof

$$R(x,a_1) \equiv \frac{f(x)-f(a_1)}{x-a_1}$$

is taken to be $F(x)$, with the convention that $F(a_1)$ is $f'(a_1)$. Thus, my opinion that "in order that Pompeiu's proof be valid, *something in addition to the existence of $f'(x)$* must be postulated" was not entirely unjustifiable.

This opinion seems to be in conformity with the view held at one time by Professor Pompeiu himself. For, in a paper [2] published after the publication of the paper containing the proof of § 15, Professor Pompeiu says: "the proof which we have given rests on a property of continuous functions. Now this property has been extended in the present paper to generalized continuous functions and this suffices to make our proof applicable to the general case. In fact, let us consider the ratio

$$R(x, a) = \frac{F(x)-F(a)}{x-a}$$

and complete its definition by putting

$$R(a, a) = F'(a).$$

$F'(x)$ may be infinite at the point $x=a$, but in this case, $R(x, a)$ is continuous (in the extended sense) at the point a."

[1] Dini's *Fondamenti per la teorica delle funzioni di variabili reali*, p. 53.

[2] "Sur le théorème des accroissements finis" Deuxieme Note, *Annales scientifiques de l' Universite de Jassy*, 1906.

APPENDIX B.

On the Various Forms of the Remainder in Taylor's Series.

The theorem giving the remainder after n terms in Taylor's series for $f(x+h)$ is so intimately connected with the mean-value theorem that it has been called by some writers the generalized mean-value theorem. For this reason, we propose to give in this Appendix first a list of the various forms of the remainder and then brief historical remarks [1] on them one after the other.

List of the Various Forms.

I. $\dfrac{h^n}{n!} f^{(n)}(x+\theta h),\ 0<\theta<1.$

[Lagrange, *Leçons sur le calcul des fonctions*, 1799.]

II. $\dfrac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(x+\theta h),\ 0<\theta<1.$

[Cauchy, *Résumé des lecons sur le calcul infinitésimal*, 1823; also *Oeuvres*, (2) T. IV, p. 260.]

III. $\dfrac{h^n}{p(n-1)!}(1-\theta)^{n-p}. f^{(n)}(x+\theta h),\ 0<\theta<1.$

[Schlömilch and Roche; Schlömilch's *Handbuch d. Diff.—u. Integral—Rechnung*, 1847; Roche's paper in *Liouville's Journal*, (2) T. 3, (1858), p. 271.]

IV. $\dfrac{\psi(h)-\psi(0)}{\psi'\{(1-\theta)h\}}\cdot\dfrac{(1-\theta)^{n-1}h^{n-1}}{(n-1)!}\ f^{(n)}(x+\theta h),\ 0<\theta<1,$

$\psi(t)$ being any continuous function with non-vanishing $\psi'(t)$ for $0<t<h$.

[Schlömilch, *Ibid*, Art. 35; see also Stolz's *Grundzüge*, Bd. 1, p. 95.]

[1] For these remarks we are indebted to a great extent to Professor A. Pringsheim's paper, "Zur Geschichte des Taylorschen Lehrsatzes" (*Bibliotheca Mathematica*, 1900, pp. 433-467).

V. $\left\{\phi(x+h)-\phi(x)-\dots-\phi^{(q)}(x).\frac{h^q}{q!}\right\}.\frac{q!(h-\theta h)^{n-q-1}}{(n-1)!}.\frac{f^{(n)}(x+\theta h)}{\phi^{(q+1)}(x+\theta h)}$,

$0<\theta<1$, $\phi(t)$ being any function which is continuous with its first $q+1$ differential co-efficients for $x \leq t \leq x+h$ with non-vanishing $\phi^{(q+1)}(t)$.

[Roche, *Comptes Rendus*, T. 58, 1864, p. 380.]

VI. $\frac{h^n}{n!}\{f^{(n)}(x)+\epsilon\}$, ϵ being a quantity which tends to 0 with h.

[Peano, Genocchi—Peano's *Calcolo Differenziale*, 1884, p. xix.]

VII. $\int_0^h dh_1\, dh_2 \dots \int_0^{h_{n-1}} f^{(n)}(x+h_n)dh_n$.

[Lacroix, *Traité élémentaire de calcul differentiel et de calcul intégral*, 2nd edition, 1806, p. 598.]

VIII. $\frac{h^n}{(n-1)!}\int_0^1 f^{(n)}(x+h-hz).\, z^{n-1}dz$.

[Lacroix, *Ibid*, p. 599.]

IX. $\frac{1}{(n-1)!}\int_0^h z^{n-1}f^{(n)}(x+h-z)dz$.

[Laplace, *Théorie Analytique des Probabilités*, No. 44, 1812; Also *Oeuvres*, t. VII, p. 179.]

X. $\frac{1}{(n-1)!}\int_0^h (h-t)^{n-1}f^{(n)}(x+t)dt$.

[Jordan, *Cours d'analyse*, T. 1, 2nd edition, 1893, p. 245.]

Brief Historical Remarks on the Forms.

I. This form of the remainder was certainly first given by Lagrange. It is found in his *Leçons sur le calcul des fonctions* delivered at the E'cole polytechnique in 1799 and published first in 1801 in the *Recueil des leçons de l'école normale* and again printed in 1804 in the Journal *de l'école polytechnique*. It may, however, be traced in his *Théorie des fonctions analytiques* published in 1797, as, after obtaining the remainder in the form of a definite integral, he gives what may be considered to be the equivalent of I. But, in the interest of truth, it should be stated that

in the *Théorie* Lagrange uses not the general index n in his working but considers the cases of $n=1, 2, 3$ and contents himself by adding that the working will do for any n.

II. This form was certainly first given by Cauchy who appears to have utilized for its derivation the remainder in the form I.

III-IV. It is a fact, as pointed by Schlömilch in his "Extrait d'une lettre de M. O. Schlömilch a M. Liouville" [*Liouville's Journal* (2) T. 3, 1858, pp. 384-385] that as early as 1847-1848 he had given in his *Handbuch der Differential u. Integral—Rechnung* the following form of the remainder:

$$\frac{\psi(h)-\psi(0)}{\psi'\{(1-\theta)h\}} \cdot \frac{(1-\theta)^{n-1}h^{n-1}}{(n-1)!} f^{(n)}(x+\theta h),$$

where ψ is an arbitrary function subject to the conditions that $\psi(t)$ together with $\psi'(t)$ is finite and continuous between the limits $x, x+h$ and that $\psi'(t)$ does not change its sign between those limits.

Schlömilch pointed out that, for $\psi(h)=h^p$, his form gives III, which was first *explicitly* published by E. Roche in the same volume of *Liouville's Journal* on pp. 271-272.

It may be added that, whilst the method used by Schlömilch is based on Cauchy's generalization of the mean-value theorem, Roche deduces his result from the form

$$\frac{1}{(n-1)!} \int_0^h t^{n-1} f^{(n)}(x+h-t)dt$$

of the remainder.

V. The following is a translation of Roche's paper[1] almost word for word:

"One may generalise the formula which makes known the finite ratio of two functions

$$\frac{F(a+h)-F(a)}{\Phi(a+h)-\Phi(a)} = \frac{F'(a+\theta h)}{\Phi'(a+\theta h)}, \tag{1}$$

where F and Φ, F′ and Φ' are supposed to be continuous and $\Phi'(x)$ to be a function which never vanishes between a and $a+h$. Let then in fact

$$F(x)=f(a+h)-f(x)-(a+h-x)f'(x)-\cdots\cdots-\frac{(a+h-x)^n}{n!} f^{(n)}(x),$$

$$\Phi(x)=\phi(a+h)-\phi(x)-(a+h-x)\phi'(x)-\cdots\cdots-\frac{(a+h-x)^q}{q!} \phi^{(q)}(x),$$

[1] It seems that the results of Roche were rediscovered by Mahajani (see his paper "A general form of the remainder in Taylor's theorem" in *Messenger of Math.*, Vol. 52, 1923, pp. 78-80).

whence

$$F'(x) = -\frac{(a+h-x)^n}{n!}\, f^{(n+1)}(x),\quad \Phi'(x) = -\frac{(a+h-x)^q}{q!}\, \phi^{(q+1)}(x).$$

Let us assume that the functions f, ϕ and their differential co-efficients remain finite and continuous in the interval from a to $a+h$, and that $\phi^{(q+1)}(x)$ is not zero; the same conditions are fulfilled by the functions F and Φ, and one may apply to them the formula (1) which reduces now in this case to

$$\frac{F(a)}{\Phi(a)} = \frac{F'(a+\theta h)}{\Phi'(a+\theta h)},$$

because

$$F(a+h)=0, \quad \Phi(a+h)=0.$$

Hence follows at once

$$\frac{f(a+h)-f(a)-hf'(a)-\cdots\cdots-\dfrac{h^n}{n!}f^{(n)}(a)}{\phi(a+h)-\phi(a)-h\phi'(a)-\cdots\cdots-\dfrac{h^q}{q!}\phi^{(q)}(a)} = \frac{q!}{n!}(h-\theta h)^{n-q}\cdot\frac{f^{(n+1)}(a+\theta h)}{\phi^{(q+1)}(a+\theta h)}, \qquad (2)$$

a relation of which the formula (1) is a particular case.

If now one puts

$$f(a+h)-f(a)-hf'(a)-\cdots\cdots-\frac{h^n}{n!}f^{(n)}(a) = R_n,$$

the equation (2) may be written

$$R_n = \left\{\phi(a+h)-\phi(a)-h\phi'(a)-\cdots\cdots-\frac{h^q}{q!}\phi^{(q)}(a)\right\}\cdot\frac{q!}{n!}(h-\theta h)^{n-q}\,\frac{f^{(n+1)}(a+\theta h)}{\phi^{q+1}(a+\theta h)}. \quad (3)$$

By giving to the arbitrary function ϕ such forms as one pleases (satisfying the conditions enunciated above), one shall have all the expressions for the remainder in Taylor's series. The equation (3) is therefore the general form of this remainder.

For example, if one puts

$$\phi(x) = (x-a)^{p+1},$$

one finds

$$R_n = \frac{q!}{n!}\,\frac{(1-\theta)^{n-q}}{\theta^{p-q}}\cdot\frac{h^{n+1}}{(p+1)p\ldots(p-q+1)}\,f^{(n+1)}(a+\theta h).$$

where p and q are undetermined; but q ought to be integral and less than the positive number $(p+1)$. One may thus obtain very well new expressions for the remainder.

In particular, for $p=q$,

$$R_n = \frac{h^{n+1}(1-\theta)^{n-p}}{n!(p+1)} f^{(n+1)}(a+\theta h),$$

a formula which I have given [1] for representing at once the two usual forms. In fact, it reproduces for $p=n$ and $p=0$, the ordinary remainder and that of Cauchy.

When one puts $q=0$ in the general expression (3) one finds the formula of Schlömilch,

$$R_n = \frac{\phi(a+h)-\phi(a)}{\phi'(a+\theta h)} \frac{h^n(1-\theta)^n}{n!} f^{(n+1)}(a+\theta h).$$

Finally, if in the interval from a to $a+h$, $f^{(n+1)}(x)$ is not anywhere zero, in other words, if in this interval $f^{(n)}(x)$ varies always in one sense, one may take $\phi(x)=f^{(n)}(x)$, $\phi'(x)=f^{(n+1)}(x)$, and then

$$R_n = \frac{h^n(1-\theta)^n}{n!} \{f^{(n)}(a+h)-f^{(n)}(a)\},$$

which furnishes a superior limit for the remainder independent of θ."

VI. In Genocchi's *Calcolo Differenziale e Principii di Calcolo Integrale publicato con aggiunte dal Dr. Giuseppe Peano* (1884), appear certain notes before the actual text. These notes are by Prof. Peano and are paged from i to xxxii. On page xix among the remarks on Art. 67 (Serie di Taylor) of the text, appear the following lines which, because of their historical importance, are reproduced here almost word for word in English translation:

"The formula of Taylor may be enunciated in the following manner:

If there exist the 1st, 2nd,...nth differential co-efficients of $f(x)$ for $x=x_0$, then

$$f(x_0+h)=f(x_0)+hf'(x_0)+\cdots+\frac{h^{(n-1)}}{(n+1)!} f^{(n-1)}(x_0)+\frac{h^n}{n!} \{f^{(n)}(x_0)+\epsilon\},$$

where ϵ is a quantity which has for its limit zero with h tending to 0. If we put $n=1$, then we have the formula which serves for the definition

[1] *Journal de M. Liouville*, 1858.

of the differential co-efficient. Of course, if $f(x)$ has the nth differential co-efficient for $x=x_0$, then it ought to have the preceding differential co-efficients also in the neighbourhood of x_0; but about the nth differential co-efficient it is not necessary to suppose either the existence or the continuity in the neighbourhood of x_0."

As stated on p. 56 of this book, the form VI is also given in Dini's *Calcolo Infinitesimale*, t. 1 (1907) which practically contains the subject-matter of Dini's lectures of much earlier years. It is, therefore, unfortunate that recently two books[1] have come out in English in which the remainder VI is prominently described as "Young's form" of the remainder. No doubt this mistake is due to the ignorance of the authors of the works of Peano and Dini, and may be excused, specially in view of the fact that so well-informed a mathematician as Professor W. H. Young gives the remainder in his book *Fundamental Theorems of the Differential Calculus* (1910) without mentioning Dini or Peano.

VII. Lacroix does not claim to be the discoverer of this form but attributes the discovery to D'Alembert, as is evidenced by his preliminary remark[2] "See here how D'Alembert has obtained and demonstrated at the same time the theorem of Taylor" (*Recherches sur differens points importans du system du monde*, t. ?, 1750, p. 50).

That Lacroix is wrong is the opinion of Professor Pringsheim[3] who says: "A provisionally given derivation of Taylor's series by D' Alembert shortly before the publication of Euler's Differential Calculus is based on the (obviously inaccurate) relation[4]

$$f(x+h)=f(x)+\int f'(x+h)dh$$

and its repeated application to $f'(x+h)$, $f''(x+h)$, etc. With the repetition n times of the process of transformation would one attain to an expression for the remainder in Taylor's expression in the form of a repeated integral with n integrations: of such a possibility has D'Alembert *not even once explicitly said anything*."

[1] Phillips, *A Course of Analysis*, 1930;
Mahajani, *Elementary Lessons in Analysis*, 1930.

[2] See Art. 384 of Lacroix's book.

[3] *L.c.*, p. 439.

[4] The correct relation is $f(x+h)=f(x)+\int_0^h f'(x+h)dh$.

VIII. (*a*) Although it is true that Lagrange was the first to give the remainder in the form of a definite integral, his form is fairly complicated and from it VIII can be derived only by a cumbersome process.

In fact Lagrange's statement giving his form of the remainder is included in the following [1] :—

Let

$$f(x+h)=f(x)+f'(x).\frac{h}{1!}+\cdots+f^{(n-1)}(x).\frac{h^{n-1}}{(n-1)!}+r_n(x,h).h^n,$$

then substitute $(x-h)$ for x and put

$$r_n(x-h,h)=q_n(x,h).$$

Thus

$$f(x)=f(x-h)+f'(x-h).\frac{h}{1!}+\cdots+f^{(n-1)}(x-h).\frac{h^{n-1}}{(n-1)!}+q_n(x,h).h^n. \qquad (1)$$

Put now

$$h=xz,\ q_n(x,xz).\ z^n=p_n(x,z),$$

then (1) gives

$$f(x)=f(x-xz)+f'(x-xz).\frac{xz}{1!}+\cdots+f^{(n-1)}(x-xz).\frac{x^{n-1}z^{n-1}}{(n-1)!}+p(x,z)x^n. \qquad (2)$$

Hence by partial differentiation with respect to z,

$$0=-f^{(n)}(x-xz).\frac{x^nz^{n-1}}{(n-1)!}+\frac{\partial p_n}{\partial z}.x^n,$$

i.e., $\dfrac{\partial p_n}{\partial z}=f^{(n)}(x-xz).\dfrac{z^{n-1}}{(n-1)!}$.

Now, from (2), we have $p_n(x,0)=0$. Therefore, finally,

$$p_n(x,z)=\frac{1}{(n-1)!}\int_0^z y^{n-1}f^{(n)}(x-xy)dy. \qquad (3)$$

If in (2) and (3) we write $\dfrac{z}{x}$ for z, we have

$$p_n\left(x,\frac{z}{x}\right)=\frac{x^n}{(n-1)!}\int_0^z y^{n-1}f^{(n)}(x-xy)dy$$

$$=\frac{1}{(n-1)!}\int_0^z t^{n-1}f^{(n)}(x-t)dt,$$

which is practically the same as any of the forms VIII—X.

[1] This is taken almost word for word from Pringsheim's paper (*l.c.*, pp. 441-442).

(b) Lacroix deduces VIII from VII by using the result

$$\int_0^{h} {}^{(n)} H(h)dh^n = \frac{1}{(n-1)!} \int_0^{h} (h-t)^{n-1} H(t)dt$$

(c) Laplace's derivation [1] of his form is substantially reproduced by Cauchy in his *Resumé* (*Lecon*, 36) and is as follows:—

"One has by taking the integral from $z=0$,

$$\int dz\phi'(x-z) = \phi(x) - \phi(x-z),$$

$$\int dz\phi'(x-z) = z\phi'(x-z) + \int z dz\phi''(x-z),$$

$$\int z dz\phi''(x-z) = \tfrac{1}{2}z^2\phi''(x-z) + \tfrac{1}{2}\int z^2 dz\phi'''(x-z).$$

By continuing this process, one finds generally

$$\int dz\phi'(x-z) = z\phi'(x-z) + \frac{z^2}{1.2}\phi''(x-z) + \ldots + \frac{z^n}{n!}\phi^{(n)}(x-z)$$

$$+ \int \frac{z^n dz}{n!} \phi^{(n+1)}(x-z).$$

By comparing this expression with

$$\int dz\phi'(x-z) = \phi(x) - \phi(x-z),$$

one shall have

$$\phi(x) = \phi(x-z) + z\phi'(x-z) + \frac{z^2}{2!}\phi''(x-z) + \ldots.$$

Putting $x-z=t$, the preceding equation takes the form

$$\phi(t+z) = \phi(t) + z\phi'(t) + \frac{z^2}{2!}\phi''(t) + \ldots\ldots + \frac{z^n}{n!}\phi^{(n)}(t)$$

$$+ \frac{1}{n!}\int_0^{z} z'^n dz'\phi^{(n+1)}(t+z-z')."$$

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[1] L.c.

APPENDIX C.

ADDITIONS AND CORRECTIONS.

Page 4,	Line 17:	*For*	" $f(x_0+h)\ f(x_0)$ "	*read*	" $f(x_0+h)-f(x_0)$."
Page 5,	2nd foot-note:	*For*	" 78 "	*read*	" 75."
Page 9,	Line 12:	*For*	"also nowhere"	*read*	"also generally nowhere."
Page 11,	Line 2:	*For*	"Pompieu"	*read*	"Pompeiu."
Page 11,	1st foot-note:	*For*	" theoreme "	*read*	" théoréme."
Page 19,	Line 18:	*For*	" $'(x\phi+$ "	*read*	" $\phi'(x+$."
Page 19,	last foot-note:	*For*	" $x \triangle x$."	*read*	" $x \triangle x$."
Page 22,	Line 6 from bottom:	*For*	" $F('\xi)$ "	*read*	" $F'(\xi)$."
Page 23,	3rd line from bottom:	*For*	" $\dfrac{\psi(x'-\delta)-\psi(x')}{\delta}$ "	*read*	" $\dfrac{\psi(x'-\delta)-\psi(x')}{-\delta}$ ".

Page 28, at the end: *Add as a foot-note to line 4*:

" Although, as shown in Appendix A, the statement in this sentence is correct, the example that follows is *not* to the point. Professor Pompeiu was good enough to respond to the author's request to express his opinion on the criticism and drew the author's attention to the second paper quoted on p. 97 in the 2nd foot-note. The proof, as modified in the light of this paper, is perfectly valid."

Page 30,	Line 20:	*For*	" $f'(\xi$ is "	*read*	" $f'(\xi)$ is."
Page 30,	foot-note:	*For*	" *Ibid* "	*read*	" See the foot-note on the preceding page."
Page 31,	Line 18:	*For*	" $f'\ b-0)$ "	*read*	" $f'(b-0)$."
Page 33,	Line 8 from bottom:	*For*	" ϕx "	*read*	" $\phi(x)$."
Page 44,	Line 1:	*For*	" inumerately "	*read*	" immediately."
Page 49,	Line 9:	*For*	" $2A_0A_1$ "	*read*	" $2A_0A_2$."
Page 51,	Line 3:	*For*	" $\{f^{(3)}\}^2 f^{(5)}$ "	*read*	" $\{f^{(3)}\}^3 . f^{(5)}$."
Page 51,	Line 9:	*For*	" f^5 "	*read*	" $f^{(5)}$."
Page 53,	Line 2:	*For*	" left "	*read*	" right."
Page 53,	Line 11:	*For*	" (2) "	*read*	" (3)."
Page 53,	Line 16:	*For*	" equation and "	*read*	" equation by $f''(x)$ and."
Page 54,	Line 3:	*For*	" left "	*read*	" right."

Page 54.	last line:	*For*	" $hf' x+\xi)$ "	*read*	" $+hf'(x+\xi)$."
Page 60.	7th line from bottom:	*For*	" $\dfrac{\cos\ \psi\left(\frac{h}{2}\right)}{h^2\psi'\ \left(\frac{h}{2}\right)}$ "	*read*	" $\dfrac{\cos\ \psi\left(\frac{h}{2}\right)}{2h^2\psi'\left(\frac{h}{2}\right)}$."
Page 62.	Add as foot-note to ξ_m in the last line: "ξ_m is the same as u_m."				
Page 64,	Line 10:	*For*	" difficult "	*read*	" not difficult."
Page 67.	Line 10:	*For*	" $x=\dfrac{2}{(4n+1)\pi}$ "	*read*	" $x=\dfrac{2}{(4n+1)\pi}$ "
Page 68,	Line 7:	*For*	" the maxima "	*read*	" the proper maxima."
Page 68,	2nd line from bottom:	*For*	" 33 "	*read*	" 32."
Page 74,	Line 2:	*For*	" 32 "	*read*	" 31."
Page 75,	Foot-note:	*For*	" second paper "	*read*	" second and third papers."

Page 76. *At the end of Art. 59 add:*

The subject is carefully considered in Prasad's third paper (*Bulletin of the Calcutta Mathematical Society*, Vol. XXIII, pp. 57-66). No doubt, $\frac{d\xi}{dh}$ cannot be finite and different from zero or infinite with determinate sign. But there is some doubt if $\frac{d\xi}{dh}$ can be zero at a point h for which $f'(h)$ has a cusp; Prasad's opinion is that most probably $\frac{d\xi}{dh}=0$ at such a point.

Page 80,	Line 1:	*For*	" values "	*read*	" of values."
Page 80,	Line 5 from bottom:	*For*	" $a_0\xi$ "	*read*	" a_0."
Page 83,	Line 15:	*For*	" $ce^{\alpha x}\ \beta x$"	*read*	" $ce^{\alpha x}+\beta x$."
Page 85	Line 3 from bottom:	*For*	" h'"	*read*	" h."
Page 87,	Line 4:	*For*	" $\dfrac{\partial\xi}{\partial x}$ "	*read*	" $\dfrac{\partial\xi}{\partial x_1}$."
Page 87,	Line 3 from bottom:	*For*	" $(x_2-x_1)f''(\xi)$. $\xi_1\ \xi_2$"	*read*	" $(x_2-x_1)f'''(\xi)$. $\xi_1\ \xi_2$."
		For	" $f''\xi(\xi_1-\xi_2)$ "	*read*	" $f''(\xi)(\xi_1-\xi_2)$."
Page 89,	Line 11 from bottom:	*For*	" $(Fx+h)$ "	*read*	" $F(x+h)$."
Page 90,	Line 5:	*For*	" $h^n 1$"	*read*	" h^n."